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DCLXI UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXIV 1984

A Characterization of the Growth of Analytic Functions by Means of Polynomial Approximation

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0. Abstract. The aim of this paper is to give a characterization of the growth of analytic functions in sets of the form $\{z \in C^N: \Phi_K(z) < R\}$ (where Φ_K is the Siciak extremal function of a compact K) by means of the polynomial approximation on K. The growth of a function g is examined by comparing $M(r) = \max \{|g(z)|. \Phi_K(z) = r\}$ and $\exp\left(\frac{R}{R-r}\right)^{\varrho}$, as r rises to R.

1. Introduction. Let K be a compact set in C^N and let $\|\ \|$ denote the supremum norm on K. For $n \in N$ denote by P_n the space of all polynomials from C^N to C, of degree at most n. In the sequel we always assume that the set K is L-regular, i.e. the Siciak extremal function of K (see [6], [8])

$$\Phi_K(z) = \sup\{|p(z)|^{\frac{1}{n}}: p \in P_n, \|p\| \leqslant 1, n \geqslant 1\}, \quad z \in C^N,$$

is continuous in C^N.

Given a function f defined and bounded on K, denote by t_n the n-th Čebyšev polynomial of the best approximation to f on K. It is known [6, 8] that if K is L-regular and $\limsup_{n\to+\infty} \|f-t_n\|^{\frac{1}{n}} = \frac{1}{R}$ with $1 < R < +\infty$, then there exists a function g analytic in $K_R = \{z \in C^N: \Phi_K(z) < R\}$ such that $g|_K = f$. In the case $R = +\infty$. T. Winiarski [12] has given necessary and sufficient conditions that the (entire) function g be of given growth. The aim of this paper is to consider the case $R < +\infty$.

For $R > 1$ let $\mathcal{O}(R) = \mathcal{O}_K(R)$ denote the set of all functions analytic in K_R and not continuable to any $K_{R'}$ with $R' > R$. Given a function $g \in \mathcal{O}(R)$, we put

$$M(r, g) = \sup\{|g(z)|: \Phi_K(z) = r\}, \qquad r < R.$$

1.1. DEFINITION. The quantity

$$\varrho = \limsup_{r\uparrow R} \frac{\log^+ \log^+ M(r, g)}{-\log\left(1 - \frac{r}{R}\right)}$$

is called the *order* of the function g.

1.2. DEFINITION. If ϱ, the order of the function g, is positive and finite, then the quantity

$$\sigma = \limsup_{r\uparrow R} \left(1-\frac{r}{R}\right)^{\varrho} \log^+ M(r,g)$$

is called the *type* of g.

Here we use the familiar notation $\log^+ x = \log x$, $x>1$, $\log^+ x = 0$, $x\leqslant 1$. Definition 1.1 is a generalization of the Beuermann definition of the order of an analytic function in the unit disc [1] (compare [10]).

In Section 4 the following theorem will be proved.

1.3. THEOREM. *If a function f is defined and bounded on a compact, L-regular, balanced set K, then*

(i) *f is the restriction to K of a function $g\in\mathcal{O}(R)$ of order ϱ $(0<\varrho<+\infty)$ if and only if*

$$\limsup_{n\to\infty} \frac{\log^+\log^+(||f-t_n||R^n)}{\log n^{\frac{\varrho}{\varrho+1}}} = 1\,;$$

(ii) *f is the restriction to K of a function $g\in\mathcal{O}(R)$ of order ϱ and type $\sigma(0<\varrho<+\infty$, $0<\sigma<+\infty)$ if and only if*

$$\limsup_{n\to\infty} \frac{\log^+(||f-t_n||R^n)}{(\varrho+1)\varrho^{-\frac{\varrho}{\varrho+1}}\sigma^{\frac{1}{\varrho+1}} n^{\frac{\varrho}{\varrho+1}}} = 1\,.$$

1.4. If the set K is the Cartesian product of plane sets, we can obtain a more precise characterization of the growth of analytic functions.

Let $L^{(1)}\times\ldots\times K^{(N)}$, where $K^{(1)},\ldots,K^{(N)}$ are L-regular compact subsets of $\boldsymbol{C}$. We put

$$K_{R_1,\ldots,R_N} = \{z\in\boldsymbol{C}^N\colon \Phi_K^{(j)}(z_j)<R_j,\ j=1,\ldots,N\}\,,\quad R_j>1\,,\quad j=1,\ldots,N\,.$$

For $R_j>1$ $(j=1,\ldots,N)$ let $\mathcal{O}(R_1,\ldots,R_N)$ denote the set of all functions analytic in $K_{R_1,\ldots,R_N}$ and not continuable analytically to any $K_{R_1',\ldots,R_N'}$ with $R_j'>R_j$ $(j=1,\ldots,N)$. Given a function f defined and bounded on K, let $t_{n_1,\ldots,n_N}$ denote the Čebyšev polynomial of the best approximation to f on K by polynomials of the degree at most n_j with respect to the j-th variable $(j=1,\ldots,N)$. We have the following theorem (see Theorem 7.3).

1.5. THEOREM. *The function f, defined and bounded on K, has an analytic continuation $g\in\mathcal{O}(R_1,\ldots,R_N)$ if and only if*

$$\limsup_{(n_1,\ldots,n_N)\to(\infty,\ldots,\infty)} (||f-t_{n_1,\ldots,n_N}||R_1^{n_1}\ldots R_N^{n_N})^{\frac{1}{n_1+\ldots+n_N}} = 1\,.$$

In the case $K=[-1,1]^N$ this theorem is a simple consequence of Theorem 2 of [5].

Let $g\in\mathcal{O}(R_1,\ldots,R_N)$. We put

$$M(r_1,\ldots,r_N,g) = \sup\{|g(z)|\colon \Phi_{K^{(j)}}(z_j)=r_j, j=1,\ldots,N\},\ 1<r_j<R_j,\ j=1,\ldots,N\,.$$

1.6. DEFINITION. A system of N positive numbers $(\varrho_1, \ldots, \varrho_N)$ is called the *order system* of the function g, if

$$\limsup_{(r_N, \ldots, r_N)\uparrow(R_1, \ldots, R_N)} \frac{\log^+\log^+ M(r_1, \ldots, r_N, g)}{\log\left[\left(1-\frac{r_1}{R_1}\right)^{-\varrho_1} + \ldots + \left(1-\frac{r_N}{R_N}\right)^{-\varrho_N}\right]} = 1.$$

1.7. DEFINITION. A system of N positive number $(\sigma_1, \ldots, \sigma_N)$ is called the *type system* of the function g corresponding to the order system $(\varrho_1, \ldots, \varrho_N)$, if

$$\limsup_{(r_1, \ldots, r_N)\uparrow(R_1, \ldots, R_N)} \frac{\log^+ M(r_1, \ldots, r_N, g)}{\sigma_1\left(1-\frac{r_1}{R_1}\right)^{-\varrho_1} + \ldots + \sigma_N\left(1-\frac{r_N}{R_N}\right)^{-\varrho_N}} = 1.$$

In Section 7 following theorem will be proved.

1.8. THEOREM. *Let* $K = K^{(1)} \times \ldots \times K^{(N)}$, *where* $K^{(1)}, \ldots, K^{(N)}$ *are compact subsets of* $\mathbf{C}$. *Assume that there exists a real number* $d>0$ *such that every connected component of each of the sets* $K^{(1)}, \ldots, K^{(N)}$ *has the ordinary diameter* $\geqslant d$. *If the function* f *is defined and bounded on* K, *then*

(i) f *has an analytic continuation* $g \in \mathcal{O}(R_1, \ldots, R_N)$ *with the order system* $(\varrho_1, \ldots, \varrho_N)$ *if and only if*

$$\limsup_{(n_1, \ldots, n_N)\to(\infty, \ldots, \infty)} \frac{\log^+\log^+(\|f-t_{n_1,\ldots,n_N}\| R_1^{n_1} \ldots R_N^{n_N})}{\log\left(n_1^{\frac{\varrho_1}{\varrho_1+1}} + \ldots + n_N^{\frac{\varrho_N}{\varrho_N+1}}\right)} = 1;$$

(ii) f *has analytic continuation* $g \in \mathcal{O}(R_1, \ldots, R_N)$ *with the type system* $(\sigma_1, \ldots, \sigma_N)$ *corresponding to the order system* $(\varrho_1, \ldots, \varrho_N)$ *if and only if*

$$\limsup_{(n_1, \ldots, n_N)\to(\infty, \ldots, \infty)} \frac{\log^+(\|f-t_{n_1,\ldots,n_N}\| R_1^{n_1} \ldots R_N^{n_N})}{\sum_{j=1}^{N} (\varrho_j+1)\varrho_j^{\frac{-\varrho_j}{\varrho_j+1}} \sigma_j^{\frac{1}{\varrho_j+1}} n_j^{\frac{\varrho_j}{\varrho_j+1}}} = 1.$$

1.9. In the case $N = 1$, under stronger assumptions on the set K, the inequality (2) of Lemma 5.3 has been proved by Mergelian [3] and Smirnov and Lebiediev [9]. The idea of the proof of Lemma 5.3 I owe to Professor J. Siciak, to whom I render here my thanks.

2. Order and type of the sum of a series of polynomials.

2.1. In the sequel we shall use the following properties of the Siciak extremal function of a compact set $K \subset \mathbf{C}^N$.

P1. $\Phi_K(z) \geqslant 1$, $z \in \mathscr{C}^N$, $\Phi_K(z) = 1$, $z \in K$.

P2. If $K = K^{(1)} \times \ldots \times K^{(N)}$, where $K^{(j)} \subset \mathbf{C}$ $(j = 1, \ldots, N)$, then

$$\Phi_K(z) = \max\{\Phi_{K^{(1)}}(z_1), \ldots, \Phi_{K^{(N)}}(z_N)\}.$$

P3. If $p \in P_n$, then $|p(z)| \leqslant \|p\| \Phi_K^n(z)$.

2.2. Let K be a fixed compact, L-regular set in C^N and let $(p_n)_{n\in N}$ be a sequence of polynomials such that

(i) $p_n \in P_n$, $n \in N$;

(ii) $\sum_{n=0}^{\infty} p_n \in \mathcal{O}(R)$ with $R \in (1, +\infty)$;

(iii) for every positive $r<R$ the set $\{\|p_n\| r^n \colon n \in N\}$ is bounded.

We put

$$M^*(r) = \max\{\|p_n\| r^n \colon n \in N\}, \quad 1<r<R;$$

$$p^* = \limsup_{r\uparrow R} \frac{\log^+\log^+ M^*(r)}{-\log\left(1-\dfrac{r}{R}\right)};$$

if $\varrho = \varrho(\sum_{n=0}^{\infty} p_n)$ is positive, we put

$$\sigma^* = \limsup_{r\uparrow R} \left(1-\frac{r}{R}\right)^{\varrho} \log^+ M^*(r).$$

2.3. Lemma. *Under the assumptions of* 2.2

(i) $\varrho \leqslant \varrho^*$;

(ii) *if* $\varrho \in (0, +\infty)$, *then* $\sigma \leqslant \sigma^*$,

(*ϱ being the order and σ the type of the function* $\sum_{n=0}^{\infty} p_n$).

Proof. Observe that for every positive $\delta<1$ and for every $r \in (1, R)$

$$M(r) \leqslant \frac{M^*(r^{\delta}R^{1-\delta})}{1-\left(\dfrac{r}{R}\right)^{1-\delta}}. \tag{1}$$

Indeed,

$$M(r) \leqslant \sum_{n=0}^{\infty} \sup\{|p_n(z)| \colon z \in K_r\}.$$

Property P3 of 2.1 gives

$$|p_n(z)| \leqslant \|p_n\| r^n, \quad z \in K_r,\ n \in N.$$

Then

$$M(r) \leqslant \sum_{n=0}^{\infty} \|p_n\| r^n.$$

Writing $r = r^{\delta}R^{1-\delta}\left(\dfrac{r}{R}\right)^{1-\delta}$ we obtain

$$M(r) \leqslant \sum_{n=0}^{\infty} M^*(r^{\delta}R^{1-\delta})\left(\frac{r}{R}\right)^{(1-\delta)n},$$

Hence

$$M(r) \leqslant \frac{M^*(r^\delta R^{1-\delta})}{1-\left(\frac{r}{R}\right)^{1-\delta}}$$

as asserted.

From (1) we obtain, for every $\delta < 1$,

$$\log^+ \log^+ M(r) \leqslant \log^+ \left[\log^+ M^*(r^\delta R^{1-\delta}) + \log \frac{1}{1-\left(\frac{r}{R}\right)^{1-\delta}}\right].$$

If the function $\{r \to M^*(r^\delta R^{1-\delta})$ is bounded, then $\varrho = \varrho^* = 0$. So we can assume that $M^*(r^\delta R^{1-\delta}) \to +\infty$ as $r \uparrow R$. Then, for r sufficiently close to R

$$\log M^*(r^\delta R^{1-\delta}) + \log \frac{1}{1-\left(\frac{r}{R}\right)^{1-\delta}} \leqslant \log M^*(r^\delta R^{1-\delta}) \log \frac{1}{1-\left(\frac{r}{R}\right)^{1-\delta}},$$

whence

$$\frac{\log^+ \log^+ M(r)}{-\log\left(1-\frac{r}{R}\right)} \leqslant \frac{\log\log M^*(r^\delta R^{1-\delta})}{-\log\left(1-\left(\frac{r}{R}\right)^\delta\right)} \cdot \frac{-\log\left(1-\left(\frac{r}{R}\right)^\delta\right)}{-\log\left(1-\frac{r}{R}\right)} + \frac{\log\log\left[1-\left(\frac{r}{R}\right)^{1-\delta}\right]^{-1}}{-\log\left(1-\frac{r}{R}\right)}.$$

After passing to the upper limit we get (i). The proof of (ii) proceeds on the same lines.

2.4. Remark. If the set K is balanced and $(p_n)_{n\in N}$ is a sequence of homogeneous polynomials, then by the Cauchy inequalities $M^*(r) \leqslant M(r)$, whence $\varrho^* \leqslant \varrho$ and $\sigma^* \leqslant \sigma$ provided $\varrho \in (0, +\infty)$.

2.5. Lemma. *Let $(p_n)_{n\in N}$ be a sequence of polynomials such that $p_n \in P_n$ for $n \in N$. If $\sum_{n=0}^{\infty} p_n \in \mathcal{O}(R)$ and if there exist positive constants β, n_0 and $\alpha < 1$ such that*

$$\|p_n\| \leqslant R^{-n} \exp(\beta n^\alpha), \quad n \geqslant n_0,$$

then

(i) $\varrho \leqslant \dfrac{\alpha}{1-\alpha}$;

(ii) *if* $\varrho = \dfrac{\alpha}{1-\alpha}$, *then* $\sigma \leqslant \dfrac{\varrho^\varrho}{(\varrho+1)^{\varrho+1}} \beta^{\varrho+1}$.

Proof. By the assumptions

$$\|p_n\| r^n \leqslant \left(\frac{r}{R}\right)^n \exp(\beta n^\alpha), \qquad n \geqslant n_0, \qquad r < R.$$

Thus

$$(2) \qquad \|p_n\| r^n \leqslant \sup\left\{\left(\frac{r}{R}\right)^x \exp(\beta x^\alpha):\ x \in (0, +\infty)\right\} =$$

$$= \exp\left[(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\beta^{\frac{1}{1-\alpha}}\left(\log\frac{r}{R}\right)^{\frac{\alpha}{\alpha-1}}\right], \qquad n \geqslant n_0, \qquad r < R.$$

Observe that for every $r < R$ there exists a positive integer $\nu(r)$ such that

$$M^*(r) = \|p_{\nu(r)}\| r^{\nu(r)}$$

and

$$M^*(r) > \|p_n\| r^n, \qquad n > \nu(r).$$

If $\nu(r)$ is bounded for $r < R$, then $M^*(r)$ also is bounded, hence $\varrho^* = 0$ and consequently $\varrho = 0$. So we may assume that $\nu(r) \geqslant n_0$ for r sufficiently close to R, say $r \in (r_0, R)$. Putting $n = \nu(r)$ in (2) we get

$$M^*(r) \leqslant \exp\left[(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\beta^{\frac{1}{1-\alpha}}\left(\log\frac{R}{r}\right)^{\frac{-\alpha}{1-\alpha}}\right], \qquad r_0 < r < R,$$

hence

$$\frac{\log^+\log^+ M^*(r)}{-\log\left(1-\frac{r}{R}\right)} \leqslant \frac{\log\left[(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\beta^{\frac{1}{1-\alpha}}\right]}{-\log\left(1-\frac{r}{R}\right)} + \frac{\alpha}{1-\alpha}\cdot\frac{\log\log\frac{R}{r}}{\log\left(1-\frac{r}{R}\right)}$$

and

$$\left(1-\frac{r}{R}\right)^\varrho \log^+ M^*(r) \leqslant \frac{\varrho^\varrho}{(\varrho+1)^{\varrho+1}}\beta^{\varrho+1}\left(\log\frac{R}{r}\right)^{-\varrho}\left(1-\frac{r}{R}\right)^\varrho$$

provided $\varrho = \dfrac{\alpha}{1-\alpha}$. Letting $r \to R$ and using Lemma 2.3 we obtain the inequalities (i) and (ii).

3. Best approximation and interpolation.

3.1. Let K be a compact, L-regular set in C^N. Given a function f defined and bounded on K we put for $n \in N$

$$E_n^{(1)} = E_n^{(1)}(f, K) = \|f - t_n\|,$$
$$E_n^{(2)} = E_n^{(2)}(f, K) = \|f - l_n\|,$$
$$E_{n+1}^{(3)} = E_{n+1}^{(3)}(f, K) = \|l_{n+1} - l_n\|,$$

where t_n denotes the n-th Čebyšev polynomial of the best approximation to f on K and l_n denotes the n-th Lagrange interpolation polynomial for f with nodes at extremal points of K (see [6]).

3.2. Inequalities [e.g. 12, Lemma 3.3].

$$E_n^{(1)} \leqslant E_n^{(2)} \leqslant (n_* + 2) E_n^{(1)},$$
$$E_n^{(3)} \leqslant 2(n_* + 2) E_{n-1}^{(1)}, \quad n \geqslant 1,$$

where $n_* = \binom{n+N}{n}$.

3.3. THEOREM [6]. *The function f is the restriction to K of a function from $\mathcal{O}(R)$ if and only if*

$$\limsup_{n\to\infty} (E_n^{(s)})^{\frac{1}{n}} = \frac{1}{R}; \quad s = 1, 2 \text{ or } 3.$$

3.4. LEMMA. *Let K be a compact, L-regular, balanced set in C^N. Then for every $g \in \mathcal{O}(R)$*

$$E_n^{(1)}(g|_K, K) \leqslant \frac{M(r, g)}{r^n(r-1)}, \quad 1 < r < R, \quad n \in N.$$

Proof. The required inequality follows immediately from a result of Siciak [6, p. 344, inequality (7)] and from the Cauchy inequalities.

4. The characterization of the growth of analytic functions. Let K be a compact, L-regular, balanced set in C^N and let R be a fixed number greater than 1.

4.1. THEOREM. *If a function g belonging to $\mathcal{O}(R)$ has a finite order ϱ, then*

$$\gamma_s = \limsup_{n\to\infty} \frac{\log^+ \log^+ (E_n^{(s)} R^n)}{\log n} = \frac{\varrho}{\varrho+1}, \quad s = 1, 2, 3.$$

Proof. By Inequalities 3.2 $\gamma_3 \leqslant \gamma_2 = \gamma_1$, so it is sufficient to show that $\gamma_1 \leqslant \dfrac{\varrho}{\varrho+1} \leqslant \gamma_3$.

1° $\gamma_1 \leqslant \dfrac{\varrho}{\varrho+1}$. From Definition 1.1 and from Lemma 3.4 for every $\mu > \varrho$ there exists $r_\mu > 1$ such that

$$\log^+ (E_n^{(1)} R^n) \leqslant \left(1 - \frac{r}{R}\right)^{-\mu} + \log\left(\frac{r}{R}\right)^n + \log\frac{1}{r_\mu - 1}, \quad r_\mu < r < R, \quad n \in N.$$

Putting $r = R\left[1 - \left(\dfrac{\mu}{n}\right)^{\frac{1}{\mu+1}}\right]$ we get

$$\log^+ (E_n^{(1)} R^n) \leqslant \left(\frac{n}{\mu}\right)^{\frac{1}{\mu+1}} - n\log\left[1 - \left(\frac{\mu}{n}\right)^{\frac{1}{\mu+1}}\right] - \log(r_\mu - 1), \quad n \geqslant n(\mu).$$

For every $\varepsilon>0$ and for sufficiently large n,

$$-\log\left[1-\left(\frac{\mu}{n}\right)^{\frac{1}{\mu+1}}\right]\leqslant(1+\varepsilon)\left(\frac{\mu}{n}\right)^{\frac{1}{\mu+1}}$$

and

$$-\log(r_\mu-1)\leqslant\varepsilon\mu^{\frac{1}{\mu+1}}n^{\frac{\mu}{\mu+1}}.$$

Hence

$$\frac{\log^+\log^+(E_n^{(1)}R^n)}{\log n}\leqslant\frac{\log[\mu^{-\frac{\mu}{\mu+1}}+(1+2\varepsilon)\mu^{\frac{1}{\mu+1}}]}{\log n}+\frac{\mu}{\mu+1}.$$

Letting $n\to\infty$ we get the inequality $\gamma_1\leqslant\frac{\mu}{\mu+1}$. By the arbitrariness of $\mu>\varrho$ we obtain

$$\gamma_1\leqslant\frac{\varrho}{\varrho+1}.$$

2° $\frac{\varrho}{\varrho+1}\leqslant\gamma_3$. If $\gamma_3=1$, then the inequality is evidently true. Assume that $\gamma_3<1$. Then for every $\alpha\in(\gamma_3,1)$

$$\frac{\log^+\log^+(E_n^{(3)}R^n)}{\log n}\leqslant\alpha$$

provided n is sufficiently large. Hence

$$E_n^{(3)}R^n\leqslant\exp(n^\alpha).$$

Using Lemma 2.5 for polynomials $p_n=l_n-l_{n-1}$, $n\geqslant1$, $p_0=l_0$, we obtain that $\frac{\varrho}{\varrho+1}\leqslant\alpha$. Owing to the arbitrariness of $\alpha>\gamma_3$

$$\frac{\varrho}{\varrho+1}\leqslant\gamma_3.$$

4.2. Theorem. *If a function g belonging to $\mathcal{O}(R)$ has a positive finite order ϱ and a finite type σ, then*

$$\lambda_s=\limsup_{n\to\infty}\frac{\log^+(E_n^{(s)}R^n)}{n^{\frac{\varrho}{\varrho+1}}}=[\chi(\varrho)\sigma]^{\frac{1}{\varrho+1}},\qquad s=1,2,3;$$

where $\chi(\varrho)=\frac{(\varrho+1)^{\varrho+1}}{\varrho^\varrho}$.

Proof. By Inequalities 3.2 $\lambda_3\leqslant\lambda_2=\lambda_1$, so it suffices to show that $\lambda_1\leqslant(\chi(\varrho)\sigma)^{\frac{1}{\varrho+1}}\leqslant\lambda_3$.

1° By Definition 1.2 for every $\omega>\sigma$ there exists $r_\omega>1$ such that

$$M(r)\leqslant\exp\left[\omega\left(1-\frac{r}{R}\right)^{-\varrho}\right],\qquad r\in(r_\omega,R).$$

Hence by Lemma 3.4

$$E_n^{(1)}R^n\leqslant\left(\frac{R}{r}\right)^n\exp\left[\omega\left(1-\frac{r}{R}\right)^{-\varrho}\right]\frac{1}{r_\omega-1},\qquad r\in(r_\omega,R),\qquad n\in N.$$

Putting

$$r=R\left[1-\left(\frac{\varrho\omega}{n}\right)^{\frac{1}{\varrho+1}}\right]$$

we get

$$\log^+(E_n^{(1)}R^n)\leqslant\omega(\omega\varrho)^{-\frac{\varrho}{\varrho+1}}n^{\frac{\varrho}{\varrho+1}}-n\log\left[1-\left(\frac{\omega\varrho}{n}\right)^{\frac{1}{\varrho+1}}\right]+\log\frac{1}{r_\omega-1},\qquad n>n(\omega).$$

Observe that for every $\varepsilon>0$ and for every sufficiently large n

$$-\log\left[1-\left(\frac{\omega\varrho}{n}\right)^{\frac{1}{\varrho+1}}\right]\leqslant(1+\varepsilon)\left(\frac{\omega\varrho}{n}\right)^{\frac{1}{\varrho+1}}$$

and

$$-\log(r_\omega-1)\leqslant\varepsilon(\omega\varrho)^{\frac{1}{\varrho+1}}n^{\frac{\varrho}{\varrho+1}}.$$

Hence

$$\frac{\log^+(E_n^{(1)}R^n)}{n^{\frac{\varrho}{\varrho+1}}}\leqslant\omega^{\frac{1}{\varrho+1}}\left[\varrho^{\frac{-\varrho}{\varrho+1}}+(1+2\varepsilon)\varrho^{\frac{1}{\varrho+1}}\right].$$

Letting $n\to\infty$, $\varepsilon\to 0$ and $\omega\to\sigma$ we get

$$\lambda_1\leqslant[\chi(\varrho)\sigma]^{\frac{1}{\varrho+1}}.$$

2° $[\chi(\varrho)\sigma]^{\frac{1}{\varrho+1}}\leqslant\lambda_3$. Suppose that $\lambda_3<[\chi(\varrho)\sigma]^{\frac{1}{\varrho+1}}$. Then there exists $\theta<\sigma$ such that $\lambda_3<[\chi(\varrho)\theta]^{\frac{1}{\varrho+1}}$, so

$$\frac{\log^+(E_n^{(3)}R^n)}{n^{\frac{\varrho}{\varrho+1}}}\leqslant[\chi(\varrho)\theta]^{\frac{1}{\varrho+1}}$$

provided n is sufficiently large. Thus

$$E_n^{(3)}\leqslant R^{-n}\exp\{[\chi(\varrho)\theta]^{\frac{1}{\varrho+1}}n^{\frac{\varrho}{\varrho+1}}\}.$$

Therefore by Lemma 2.5 $\sigma\leqslant\theta$ and we get a contradiction, because θ has been choosen less than σ. Hence $\lambda_3\leqslant[\chi(\varrho)\sigma]^{\frac{1}{\varrho+1}}$ as asserted.

The following will show how the speed of convergence to 0 of the sequences $(E_n^{(s)}(f, K))_{n\in N}$ estimates the set on which the function f can be extended analytically and determines the growth of this extension.

4.3. THEOREM. *Given a function f, defined and bounded on K, we put*

$$\alpha = \limsup_{n\to\infty} \frac{\log^+\log^+(E_n^{(1)}R^n)}{\log n}.$$

If $\alpha \in (0, 1)$, then the function $\tilde{f} = l_0 + \sum_{n=1}^{\infty}(l_n - l_{n-1})$ belongs to $\mathcal{O}(R)$, $\tilde{f}|_K = f$ and $\tilde{f}$ has the order $\varrho = \frac{\alpha}{1-\alpha}$.

Proof. For every $\mu \in (\alpha, 1)$

$$\frac{\log^+\log^+(E_n^{(1)}R^n)}{\log n} \leqslant \mu$$

provided n is sufficiently large. Hence

$$E_n^{(1)}R^n \leqslant \exp(n^\mu) \tag{3}$$

and consequently

$$\limsup_{n\to\infty}(E_n^{(1)}R^n)^{\frac{1}{n}} \leqslant 1 .$$

Since $\alpha > 0$, the sequence $(E_n^{(1)}R^n)_{n\in N}$ is unbounded, whence

$$\limsup_{n\to\infty}(E_n^{(1)}R^n)^{\frac{1}{n}} \geqslant 1 .$$

Thus

$$\limsup_{n\to\infty}(E_n^{(1)})^{\frac{1}{n}} = \frac{1}{R}.$$

So by Theorem 3.3 $\tilde{f} \in \mathcal{O}(R)$. Moreover, inequalities (3) and 3.2 give

$$\|l_n - l_{n-1}\| \leqslant R^{-n}\exp(2n^\mu)$$

provided n is sufficiently large. Hence by Lemma 2.5 and Theorem 4.1

$$\varrho = \frac{\alpha}{1-\alpha}.$$

4.4. THEOREM. *Let f be a function defined and bounded on K. If for some positive and finite ϱ*

$$\beta(\varrho) = \limsup_{n\to\infty} n^{-\frac{\varrho}{\varrho+1}}\log^+(E_n^{(1)}R^n)$$

is finite, then the function $\tilde{f} = l_0 + \sum_{n=1}^{\infty}(l_n - l_{n-1})$ *belongs to* $\mathcal{O}(R)$, $\tilde{f}|_K = f$, ϱ *is the order of* $\tilde{f}$ *and* $\sigma = \frac{\varrho^\varrho}{(\varrho+1)^{\varrho+1}}\beta(\varrho)^{\varrho+1}$ *is the type of* $\tilde{f}$.

Proof. Repeating the arguments used in the proof of Theorem 4.3 one may easily check that $\tilde{f} \in \mathcal{O}(R)$. In order to estimate the growth of $\tilde{f}$ take any $\tau > \beta(\varrho)$. Then

$$n^{-\frac{\varrho}{\varrho+1}}\log^+(E_n^{(1)}R^n) \leqslant \tau, \qquad n > n(\tau).$$

Hence the inequality

$$E_n^{(1)} \leqslant R^{-n}\exp(\tau n^{\frac{\varrho}{\varrho+1}})$$

together with Inequalities 3.2 imply

$$\|l_n - l_{n-1}\| \leqslant R^{-n}\exp(2\tau n^{\frac{\varrho}{\varrho+1}})$$

provided n is sufficiently large. Hence by Lemma 2.5 $\tilde{\varrho}$, the order of $\tilde{f}$, satisfies $\tilde{\varrho} \leqslant \varrho$. Suppose that $\tilde{\varrho} < \varrho$. Then, oving to Theorem 4.1, for every $\varrho' \in (\tilde{\varrho}, \varrho)$

$$\frac{\log^+\log^+(E_n^{(1)}R^n)}{\log n} \leqslant \frac{\varrho'}{\varrho'+1}, \qquad n > n'(\varrho').$$

Thus

$$\limsup_{n\to\infty} n^{-\frac{\varrho'}{\varrho'+1}}\log^+(E_n^{(1)}R^n) \leqslant 1.$$

On the other hand

$$\limsup_{n\to\infty} n^{-\frac{\varrho'}{\varrho'+1}}\log^+(E_n^{(1)}R^n) = \limsup_{n\to\infty} n^{-\frac{\varrho}{\varrho+1}}\log^+(E_n^{(1)}R^n)\, n^{\frac{\varrho}{\varrho+1}-\frac{\varrho'}{\varrho'+1}} = +\infty.$$

So we get a contradiction, whence $\tilde{\varrho} = \varrho$. Moreover, by Lemma 2.5 and Theorem 4.2

$$\sigma = \frac{\varrho^\varrho}{(\varrho+1)^{\varrho+1}}\beta(\varrho)^{\varrho+1}.$$

4.5. Remark. It is obvious that in Theorems 4.3 and 4.4 $E_n^{(1)}$ may be replaced by $E_n^{(2)}$ or $E_n^{(3)}$.

5. Order and type systems.

5.1. In Sections 5 through 7 we consider an approximation and interpolation on $K = K^{(1)} \times \ldots \times K^{(N)}$, where $K^{(1)}, \ldots, K^{(N)}$ are compact L-regular sets in $\mathbf{C}$.

Given $r = (r_1, \ldots, r_N) \in (1, +\infty)^N$ and $j \in N$, $1 \leqslant j \leqslant N$, we put

$$\Phi_j = \Phi_{K(j)},$$

$$K_{r_j}^{(j)} = \{\zeta \in C\colon \Phi_j(\zeta) < r_j\},$$

$$C_{r_j}^{(j)} = \{\zeta \in C\colon \Phi_j(\zeta) = r_j\},$$

$$K_r = K_{r_1}^{(1)} \times \ldots \times K_{r_N}^{(N)}.$$

For $n = (n_1, \ldots, n_N) \in N^N$ let P_n denote the space of all polynomials from C^N to C of degree at most n_j with respect to the j-th variable ($j = 1, \ldots, N$). Moreover we shall use the following notations:
if

$$x = (x_1, \ldots, x_N) \in R^N,\ y = (y_1, \ldots, y_N) \in R^N,\ \xi \in R,$$

then

$$|x| = |x_1| + \ldots + |x_N|;$$

$$x \cdot y = (x_1 y_1, \ldots, x_N y_N);$$

$$\frac{x}{y} = \left(\frac{x_1}{y_1}, \ldots, \frac{x_N}{y_N}\right); \quad y_j \neq 0,\ j = 1, \ldots, N;$$

$$x^y = x_1^{y_1} \ldots x_N^{y_N},\ x^\xi = (x_1^\xi, \ldots, x_N^\xi), \quad x_j > 0,\ j = 1, \ldots, N;$$

$$\hat{\xi} = (\xi, \ldots, \xi) \in R^N;$$

$$\hat{\infty} = \underbrace{(\infty, \ldots, \infty)}_{N \text{ times}};$$

$$\chi(\xi) = \frac{(\xi+1)^{\xi+1}}{\xi^\xi}, \quad \xi > 0;$$

$$x < y \overset{\text{df}}{\Leftrightarrow} x_j < y_j,\ j = 1, \ldots, N;\ x > y \overset{\text{df}}{\Leftrightarrow} y < x.$$

5.2. The order and type systems of a function from $\mathcal{O}(R)$ may be defined as elements of some surfaces in $(\mathcal{O}, +\infty)^N$ (compare [4, p. 174]). Given $R = (R_1, \ldots, R_N) > \hat{1}$ and $g \in \mathcal{O}(R)$ let $\mathcal{R} = \mathcal{R}(g)$ denote the set of all $\mu \in (0, +\infty)^N$ for which there exists $r(\mu) \in (1, R_1) \times \ldots \times (1, R_N)$ such that

$$\log^+ M(r) \leqslant \sum_{j=1}^{N} \left(1 - \frac{r_j}{R_j}\right)^{-\mu_j}, \quad r(\mu) < r < R.$$

Analogously to the case of entire functions [4] one can readily show that

the system of N positive numbers $(\varrho_1, \ldots, \varrho_N)$ *is the order system of a function g if and only if* $(\varrho_1, \ldots, \varrho_N) \in \partial\mathcal{R}(g)$.