

Some Properties of Functional-differential Equations on Invariant Sets

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1. H. Brezis ([1]) proved that for ordinary differential equations the Nagumo boundary condition is equivalent to the requirement that the solution of the Cauchy problem starting from a given closed set remains in this set. He considered a case in which right-hand side of the equation fulfilled the Lipschitz condition. A similar problem with a continuous right-hand side has been considered by Hartman ([3]). The aim of this paper is to extend these results to the case of functional-differential equations. In addition we shall show that functional-differential equations with a continuous right-hand side satisfying the Nagumo condition generate a generic global flow. It should be remembered that the Nagumo condition for functional-differential equations has previously been used by S. Leela, V. Lakshmikantham, and V. Moauro [4].

2. Let Ω be an open subset of the Euclidean space R^n and F be a closed subset of Ω . Let $A = C([-h, 0], \Omega)$ and $B = C([-h, 0], F)$. For $\varphi \in A$ set

$$\|\varphi\| = \sup\{|\varphi(\tau)| \mid \tau \in [-h, 0]\}.$$

Let $K(\varphi, \delta) = \{\psi \in A \mid \|\psi - \varphi\| < \delta\}$. Denote by x_t the element in A given by $x_t(\tau) = x(t+\tau)$, $-h \leq \tau \leq 0$, $x(\cdot)$ a continuous function of t . For a continuous and bounded function $f: A \rightarrow R^n$ and $(t_0, \varphi_0) \in R^+ \times B$, $R^+ = [0, +\infty)$ we consider the Cauchy problem

$$(1) \quad \begin{cases} x' = f(x_t), \\ x_{t_0} = \varphi_0. \end{cases}$$

DEFINITION. For a given $t_0 \in R^+$, $\varphi_0 \in A$ we say $x(\cdot; t_0, \varphi_0)$ is a solution of the equation (1) through (t_0, φ_0) , if there is a $b > 0$ such that $x(\cdot; t_0, \varphi_0)$ satisfies the equation (1) on $[t_0 - h, t_0 + b]$ and $x_{t_0} = \varphi_0$.

THEOREM 1. Let $\Omega \subset R^n$ be an open set and $F \subset \Omega$ be a closed set. Let $A = C([-h, 0], \Omega)$. Then for every continuous and bounded function $f: A \rightarrow R^n$ the following conditions are equivalent:

I. for every $(t_0, \varphi_0) \in R^+ \times B$ (1) has at least one solution such that $x(t; t_0, \varphi_0) \in F$ for $t \in [t_0 - h, t_0 + b]$,

II. for every $(t, \varphi) \in R^+ \times B$

$$(2) \quad \lim_{k \rightarrow 0} d(\varphi(0) + kf(\varphi), F)/k = 0.$$

Proof. First we prove that II implies I. Fix for the moment $(t_0, \varphi_0) \in R^+ \times B$. Let M be a constant such that $|f(\varphi)| < M$ for $\varphi \in A$. Since A is open, there is $\delta_1 > 0$ such that $K(\varphi_0, \delta_1) \subset A$. Clearly, for any $t \in [t_0, t_0 + b]$, there exists a point $y^t \in F$ which satisfies

$$d(\varphi_0(0) + (t - t_0)f(\varphi_0), F) = |\varphi_0(0) + (t - t_0)f(\varphi_0) - y^t|.$$

Denote $\varphi_0(0) = x_0$. It is easy to see that

$$(3) \quad |y^t - x(t; t_0, \varphi_0)| \\ \leq |x(t; t_0, \varphi_0) - x_0 - (t - t_0)f(\varphi_0)| + d(x_0 + (t - t_0)f(\varphi_0), F) = o(t - t_0);$$

$$(4) \quad \frac{y^t - x_0}{t - t_0} = \frac{x(t; t_0, \varphi_0) - x_0}{t - t_0} + \frac{y^t - x(t; t_0, \varphi_0)}{t - t_0} = f(\varphi_0) + o(t - t_0).$$

Take $\varepsilon > 0$. By (4) there is a constant $s_1 > 0$ such that

$$\left| \frac{y^t - x_0}{t - t_0} \right| < |f(\varphi_0)| + \frac{\varepsilon}{2} \quad (t_0 \leq t \leq t_0 + s_1).$$

Since f is continuous, there exists $\delta_2 > 0$ such that

$$(5) \quad |f(\varphi_0) - f(\psi)| < \frac{\varepsilon}{2} \quad \text{for } \psi \in A, \quad \|\varphi_0 - \psi\| < \delta_2.$$

Let $\delta = \min(\delta_1, \delta_2)$. From the uniform continuity of φ_0 in $[-h, 0]$ there follows the existence of $s_2 > 0$ such that

$$(6) \quad |\varphi_0(\tau_1) - \varphi_0(\tau_2)| < \frac{\delta}{2} \quad \text{if } |\tau_1 - \tau_2| \leq s_2, \quad \tau_1, \tau_2 \in [-h, 0].$$

By (3) and the continuity of solution $x(\cdot; t_0, \varphi_0)$ there are s_3 and s_4 such that

$$|y^t - x(t; t_0, \varphi_0)| < \frac{\delta}{4} \quad \text{as } t_0 \leq t \leq t_0 + s_3,$$

$$|x(t; t_0, \varphi_0) - x_0| < \frac{\delta}{4} \quad \text{as } t_0 \leq t \leq t_0 + s_4.$$

Put $s = \min(s_1, s_2, s_3, s_4)$. Then for $t \in [t_0, t_0 + s]$ we have

$$(7) \quad |y^t - x_0| \leq |y^t - x(t; t_0, \varphi_0)| + |x(t; t_0, \varphi_0) - x_0| < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}.$$

Let y^{s+t_0} be a point in F such that

$$d(\varphi_0(0) + sf(\varphi_0), F) = |\varphi_0(0) + sf(\varphi_0) - y^{s+t_0}|.$$

Let us consider the function $z: [t_0 - h, t_0 + s] \rightarrow R^n$, defined by

$$(8) \quad z(\tau) = z(\tau; t_0, x_0, s) = \begin{cases} \varphi_0(\tau - t_0) & \text{for } t_0 - h \leq \tau \leq t_0, \\ x_0 + (\tau - t_0) \frac{y^{s+t_0} - x_0}{s} & \text{for } t_0 \leq \tau \leq t_0 + s. \end{cases}$$

Note that for $t \in [t_0, t_0 + s]$ by (6) and (7) we have

$$\begin{aligned} \|z_t - \varphi_0\| &= \max(\|\varphi(\cdot) + t - t_0 - \varphi_0(\cdot)\|_{[t-h, t_0-t]}, \|\varphi_0(\cdot) - z_t(\cdot)\|_{[t_0-t, t_0]}) \\ &\leq \max\left(\frac{\delta}{2}, \|\varphi_0(\cdot) - \varphi_0(0)\|_{[t_0-t, t_0]} + \|z(\cdot) - \varphi_0(0)\|_{[t_0, t_1]}\right) \leq \max\left(\frac{\delta}{2}, \frac{\delta}{2} + \frac{\delta}{2}\right) = \delta. \end{aligned}$$

So the function z_t belongs to $K(\varphi_0, \delta)$. Furthermore

$$\begin{aligned} z_{t_0} &= \varphi_0, \quad z(t_0 + s) = y^{t_0 + s} \in F, \\ |z'(t) - f(z_t)| &= \left| \frac{y^{s+t_0} - x_0}{s} - f(z_t) \right| = |0(t - t_0) + f(\varphi_0) - f(z_t)| \\ &\leq |0(t - t_0)| + |f(\varphi_0) - f(z_t)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad (t \in [t_0, t_0 + s]). \end{aligned}$$

Let Q_ε be a set of pairs (y, T) for $0 < \varepsilon \leq M$, where $t_0 < T \leq t_0 + b$, y is a function defined on $[t_0 - h, T]$ and such that the following conditions hold good:

- (i) $y_{t_0} = \varphi_0$;
- (ii) $|y(\tau) - x_0| \leq 2M|T - t_0| \leq 2Mb$, $\tau \in [t_0, T]$;
- (iii) $|y(\tau_1) - y(\tau_2)| \leq 2M|\tau_1 - \tau_2|$, $\tau_1, \tau_2 \in [t_0, T]$;
- (iv) $|y'(t) - f(y_t)| \leq \varepsilon$ for almost all $t \in [t_0, T]$.
- (v) Each subinterval of $[t_0, T]$ with length greater than or equal to ε contains a point t_1 such that $y(t_1) \in F$.

We may assume, without loss of generality, that the previously chosen constant s is less than or equal to ε . Note that the set Q_ε is non-empty, since it contains $(z, t_0 + s)$ where the function z is defined by (8). Introduce the partial ordering relation in Q_ε given by

$$(y_1, T_1) < (y_2, T_2) \Leftrightarrow T_1 < T_2 \quad \text{and} \quad y_2(t) = y_1(t), \quad t \in [t_0 - h, T_1]$$

Every chain in Q_ε possesses an upper bound in Q_ε . Thus by Kuratowski-Zorn's lemma, Q_ε contains a maximal element $(y_\varepsilon, T_\varepsilon)$. Now we shall prove that $T_\varepsilon = t_0 + b$. Assume for contradiction that $T_\varepsilon < t_0 + b$. Then (reasoning as previously), a constant $\tilde{s} > 0$ may be chosen such that function

$$\tilde{y}_\varepsilon(t) = \begin{cases} y_\varepsilon(t), & \text{for } t_0 - h \leq t \leq T_\varepsilon \\ y_\varepsilon(T_\varepsilon) + (t - T_\varepsilon) \frac{y^{s+T_\varepsilon} - y_\varepsilon(T_\varepsilon)}{\tilde{s}}, & \text{for } T_\varepsilon \leq t \leq T_\varepsilon + \tilde{s} \end{cases}$$

which is an extension of function y_ε on an interval $[T_\varepsilon, T_\varepsilon + \delta]$ satisfies (i)–(v). This contradicts the fact that $(y_\varepsilon, T_\varepsilon)$ is the maximal element of Q_ε . Consider now a sequence $\varepsilon_i \rightarrow 0$, $i \rightarrow \infty$. The functional sequence $y_{\varepsilon_i}(\cdot)$, $i = 1, 2, \dots$ of maximal elements in sets Q_{ε_i} satisfies the assumption of the Ascoli–Arzeli theorem. Thus it is uniformly convergent on $[t_0 - h, t_0 + b]$. It is not difficult to see that $x(t) = \lim_{i \rightarrow \infty} y_{\varepsilon_i}(t)$ is the solution of (1) and

$x(t) \in F$ for $t \in [t_0, t_0 + b]$, which completes the first part of the proof.

Now we prove the second part, i.e. I implies II. Let $x(\cdot; s, \varphi)$ be a solution of (1) with $t_0 = s$, $\varphi_0 = \varphi$, $(s, \varphi) \in R^+ \times B$, defined on $[s - h, s + b]$, $b > 0$. From assumption I we have for $0 < k < b$

$$\begin{aligned} \frac{d(\varphi(0) + kf(\varphi), F)}{k} &= \frac{1}{k} \inf_{y \in F} (|\varphi(0) + kf(\varphi) - y|) \\ &\leq \left| \frac{x(s+k; s, \varphi) - \varphi(0)}{k} - f(\varphi) \right| + \frac{1}{k} \inf_{y \in F} (|x(s+k; s, \varphi) - y|) \\ &= \left| \frac{x(s+k; s, \varphi) - \varphi(0)}{k} - f(\varphi) \right|. \end{aligned}$$

If $k \rightarrow 0$ the right-hand side of the last equation also converges to zero and the proof of our theorem is completed.

CONCLUSION 1. *Let $f: A \rightarrow R^n$ be a Lipschitzean function and moreover satisfy the Nagumo condition (2). Then for any $(t_0, \varphi_0) \in R^+ \times B$ the problem (1) has in $[t_0, +\infty)$ exactly one solution $x(\cdot; t_0, \varphi_0)$ which is in F for each $t \in [t_0, +\infty)$.*

Denote by M the set of all continuous and bounded functions $f: A \rightarrow R^n$ which satisfy the Nagumo condition (2). It is well known that M is a complete metric space ([5]). Furthermore the subset of all locally Lipschitzean functions is dense in M ([2]). Using conclusion 1 and repeating the arguments of [2] the following may be proved:

THEOREM 2. *Let M_0 be a set of all $f \in M$ for which there exists a residual subset B_f of B such that for every $\varphi_0 \in B_f$ the problem (1) has exactly one solution $x(\cdot; t_0, \varphi_0)$, defined on $[t_0 - h, +\infty)$ and such that $x(t; t_0, \varphi_0) \in F$ for every $t \in [t_0 - h, +\infty)$. Then M_0 is a residual subset of M .*

References

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