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On Degenerate Non-linear Parabolic Functional-differential Inequalities in Arbitrary Domains

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Introduction. A system of nonlinear, second order degenerate functional-differential inequalities of the type

$$(0.1)\quad F^i(t,x,u(t,x),u^i_t(t,x),u^i_x(t,x),u^i_{xx}(t,x),u)$$
$$>F^i(t,x,v(t,x),v^i_t(t,x),v^i_x(t,x),v^i_{xx}(t,x),v)\quad (i=1,\dots,m)$$

is considered, where

$$x=(x_1,\dots,x_n),\ u(t,x)=(u^1(t,x),\dots,u^m(t,x)),\ v(t,x)=(v^1(t,x),\dots,v^m(t,x)),$$
$$u^i_x(t,x)=(u^i_{x_1}(t,x),\dots,u^i_{x_n}(t,x)),\ v^i_x(t,x)=(v^i_{x_1}(t,x),\dots,v^i_{x_n}(t,x)),$$

$u^i_{xx}(t,x)$, $v^i_{xx}(t,x)$ are the matrices of second order derivatives with respect to x, $(t,x)\in D$ and $D\subset(t,x_1,\dots,x_n)$ is an arbitrary set satisfying adequate assumptions. The functions $F^i(t,x,z,p,q,r,w)$ $(i=1,\dots,m)$ are weakly decreasing with respect to p. The symbols $u=(u^1,\dots,u^m)$ and $v=(v^1,\dots,v^m)$ denote adequately regular functions defined in some subset of D_0, having their values in R^m, and $D_0\subset\{(t,x)\colon t<t_0+T\}$ is an arbitrary set such that $D_0\supset\bar{D}\cap\{(t,x)\colon t<t_0+T\}$. For degenerate parabolic inequalities of the type (0.1) two theorems are proved.

The results obtained are a generalization of those given by J. Szarski (see [1], Theorem 64.5; [2], Theorem 1). This development is not of a formal nature since only weak monotonicity of the functions F^i $(i=1,\dots,m)$ with respect to p is required. The proofs of the obtained theorems are patterned on those given in [1] and [2], and the idea of this paper is patterned on that introduced by M. C. Waid in [4]. For ease in reference to the theorems of J. Szarski the same notations are used as in [1], [2] and [3].

1. Notation, definitions and assumptions. For any vectors $z=(z^1,\dots,z^m)$, $\tilde{z}=(\tilde{z}^1,\dots,\tilde{z}^m)$ we write

$$z<\tilde{z}\quad \text{if } z^j<\tilde{z}^j\quad (j=1,\dots,m).$$

Assumption A. *D is an arbitrary open set in the time-space $(t, x_1, \dots, x_n)$ and its projection on the t-axis is the interval (t_0, t_0+T), $t_0+T \leqslant \infty$.*

For any set $E \subset (t, x_1, \dots, x_n)$ and any $\hat{t} \in (t_0, t_0+T]$ we put

$$E^{\hat{t}} = E \cap \{(t, x)\colon\ t \in (t_0, \hat{t}),\ x \in R^n\}\,.$$

By D_p we denote the subset of those points $(\bar{t}, \tilde{x})$ belonging to $(D \cup \partial D)^{t_0+T}$ for which there is a lower half neighbourhood

$$t < \bar{t},\ \sum_{j=1}^{n} (x_j - \tilde{x}_j)^2 + (t - \bar{t})^2 < r^2$$

contained in D. It is clear that $D \subset D_p$.

For every fixed $t \in (t_0, t_0+T)$ we put

$$S_t = \{x \in R^n\colon\ (t, x) \in D_p\}\,.$$

It is obvious that S_t is open in R^n. We denote by $\sum$ that part of $(\partial D)^{t_0+T}$ which is disjoint with D_p and by S_{t_0} the subset of ∂D contained in the plane $t = t_0$. Finally, we denote by $D_0 \subset \{(t, x)\colon\ t < t_0+T\}$ an arbitrary set such that $D_0 \supset \bar{D} \cap \{(t, x)\colon\ t < t_0+T\}$.

Assumption B. *Let $\sum_i$ $(i = 1, \dots, m)$ be a subset (possibly empty) of $\sum$. For $(t, x) \in \sum_i$ $l^i(t, x)$ is a direction such that l^i is orthogonal to the t-axis and some open segment, with one extremity at (t, x), of the straight half line from (t, x) in the direction l^i is contained in D_p.*

We say that a function $u^i\colon D_0 \to R$ has *finite t-right sided limits* in $S_{t_0} \cup (\sum \setminus \sum_i) \cup \infty$ if for every $(\bar{t}, \tilde{x}) \in S_{t_0} \cup (\sum \setminus \sum_i)$ and every $\bar{t} \in P(S_{t_0} \cup (\sum \setminus \sum_i))$, where $P(S_{t_0} \cup (\sum \setminus \sum_i))$ is the projection of $S_{t_0} \cup (\sum \setminus \sum_i)$ on the t-axis, and for each sequence $(t^\nu, x^\nu) \in D_p$ such that $t^\nu > \bar{t}$, $t^\nu \to \bar{t}$ and $(t^\nu, x^\nu) \to (\bar{t}, \tilde{x})$ or $|x^\nu| \to \infty$, the limit

$$\lim_{\nu \to \infty} u^i(t^\nu, x^\nu)$$

is finite. Obviously, this limit does not depend on the choice of the sequence (t^ν, x^ν) and will be denoted by $u^i_{(+)}(\bar{t}, \tilde{x})$ resp. $u^i_{(+)}(\bar{t}, \infty)$.

Let $C_m(D_0)$ denote the space of all functions $w = (w^1, \dots, w^m)\colon D_0 \to R^m$, such that, for every i, w^i is continuous in $D_p \cup \sum_i$ and has finite t-right sided limits in $S_{t_0} \cup \cup (\sum \setminus \sum_i) \cup \infty$.

By Y we denote a fixed subset of the space $C_m(D_0)$.

For $w, \tilde{w} \in C_m(D_0)$ and for every fixed $t < t_0+T$, we write $w \overset{t}{\leqslant} \tilde{w}$, if $w^i(\bar{t}, x) \leqslant \tilde{w}^i(\bar{t}, x)$ for $(\bar{t}, x) \in D_0$, $\bar{t} \leqslant t$ $(i = 1, \dots, m)$.

Given the sets $\sum_i$ and directions l^i $(i = 1, \dots, m)$ satisfying Assumption B, a function $u \in C_m(D_0)$ is called ***$\sum$-regular*** in D_p if, for every i, u^i_t, u^i_x, u^i_{xx} are continuous in D_p and the derivative $\dfrac{du^i}{dl^i}$ is finite in every point of $\sum_i$.

LEMMA. *Assume that for the given sets* $\sum_i$ *and the directions* l^i $(i = 1, \ldots, m)$ *satisfying Assumption* B, *for the given functions* $a^i(t, x) \geqslant 0$, *defined for* $(t, x) \in \sum_i$, *and for the given real functions* $\varphi^i(t, x, z)$, *defined for* $(t, x) \in \sum_i$, $z \in R$ *and weakly increasing with respect to* z, *the* $\sum$*-regular in* D_p *functions* u *and* v *satisfy the strong initial and boundary inequalities*

$$(1.1) \qquad u^i_{(+)}(t, x) - v^i_{(+)}(t, x) < 0$$

for $(t, x) \in S_{t_0} \cup (\sum \setminus \sum_i) \cup \infty \quad (i = 1, \ldots, m)$ *and*

$$(1.2) \qquad \varphi^i(t, x, u^i(t, x)) - \varphi^i(t, x, v^i(t, x)) < a^i(t, x) \frac{d[u^i(t, x) - v^i(t, x)]}{dl^i}$$

for $(t, x) \in \sum_i$ $(i = 1, \ldots, m)$.

Then the following assertions hold true

1) *There is* $t^* \in (t_0, t_0 + T)$, *such that*

$$(1.3) \qquad u^i(t, x) - v^i(t, x) < 0 \qquad (i = 1, \ldots, m)$$

in $D_p^{t^*}$.

2) *If for a* $\bar{t} \in (t_0, t_0 + T)$ *we have inequalities*

$$(1.4) \qquad u^i(\bar{t}, x) - v^i(\bar{t}, x) < 0 \qquad \text{for } x \in S_{\bar{t}} \ (i = 1, \ldots, m),$$

then there is a $\hat{t} \in (\bar{t}, t_0 + T)$, *such that inequalities* (1.3) *hold in*

$$D_p \cap \{(t, x) \colon \bar{t} \leqslant t < \hat{t}, x \in R^n\}.$$

Proof. Suppose 1) is not true. Then there would exist a sequence $(t^\nu, x^\nu) \in D_p$ and an index j such that $t^\nu > t_0$, $t^\nu \to t_0$, $(t^\nu, x^\nu) \to (t_0, x_0) \in S_{t_0}$, or $|x^\nu| \to \infty$, and

$$(1.5) \qquad u^j(t^\nu, x^\nu) - v^j(t^\nu, x^\nu) \geqslant 0.$$

From (1.5) follows

$$(1.6) \qquad \liminf_{\nu \to \infty} [u^j(t^\nu, x^\nu) - v^j(t^\nu, x^\nu)] \geqslant 0,$$

in contradiction with the assumption that $u(t, x) - v(t, x)$ satisfies the strong initial inequality (1.1).

Now, suppose that 2) does not hold true; then there would exist a sequence $(t^\nu, x^\nu) \in D_p$ and an index j so that $t^\nu > \bar{t}$, $t^\nu \to \bar{t}$, $(t^\nu, x^\nu) \to (\bar{t}, \tilde{x}) \in \sum \cup (\bar{t} \times S_{\bar{t}})$ or $|x^\nu| \to \infty$, and (1.5) is satisfied. Since by the assumption of 2) we have inequalities (1.4) it follows, by (1.5) and continuity of $u^j(t, x) - v^j(t, x)$ in D_p, that $(\bar{t}, \tilde{x}) \notin (\bar{t} \times S_{\bar{t}})$.

If $(\bar{t}, \tilde{x}) \in \sum \setminus \sum_j$ or $|x^\nu| \to \infty$, we obtain (1.6) (like in 1) in contradiction with the assumption that $u(t, x) - v(t, x)$ satisfies the strong boundary inequality (1.1). If $(\bar{t}, \tilde{x}) \in \sum_j$, then by (1.5) and continuity of $u^j(t, x) - v^j(t, x)$ in $\sum_j$, we have

$$(1.7) \qquad u^j(\bar{t}, \tilde{x}) - v^j(\bar{t}, \tilde{x}) \geqslant 0$$

and according to the monotonicity of $\varphi^j(t, x, z)$ with respect to z, we obtain

$$(1.8) \qquad \varphi^j(\bar{t}, \tilde{x}, u^j(\bar{t}, \tilde{x})) - \varphi^j(\bar{t}, \tilde{x}, v^j(\bar{t}, \tilde{x})) \geqslant 0.$$

On the other hand, by (1.7) and by the inequality $u^j(\tilde{t}, x) - v^j(\tilde{t}, x) < 0$, satisfied for $x \in S_{\tilde{t}}$, we have

$$\frac{d[u^j(t, x) - v^j(t, x)]}{dl^j}\bigg|_{(\tilde{t}, \tilde{x})} \leqslant 0\,. \tag{1.9}$$

From (1.8), (1.9), and $a^j(t, x)$ being non-negative, it follows that

$$\varphi^j(\tilde{t}, \tilde{x}, u^j(\tilde{t}, \tilde{x})) - \varphi^j(\tilde{t}, \tilde{x}, v^j(\tilde{t}, \tilde{x})) \geqslant a^j(\tilde{t}, \tilde{x})\, \frac{d[u^j(t, x) - v^j(t, x)]}{dl^j}\bigg|_{(\tilde{t}, \tilde{x})},$$

in contradiction with the strong boundary inequality (1.2).

Let $F^i(t, x, z, p, q, r, w)$ $(i = 1, \ldots, m)$, where $q = (q_1, \ldots, q_n)$ and $r = (r_{jk})_{j,k=1}^n$ is an $n \times n$ real, symmetric matrix, be real functions defined for $(t, x) \in D_p$, $z \in R^m$, $p \in R$, $q \in R^n$, $r \in R^{n \times n}$ and $w \in Y$.

The $\sum$-regular functions u and v in D_p belonging to Y are called *$\sum$-regular solutions* of the system

$$\begin{aligned} F^i(t, x, u(t, x), u_t^i(t, x), u_x^i(t, x), u_{xx}^i(t, x), u) \\ > F^i(t, x, v(t, x), v_t^i(t, x), v_x^i(t, x), v_{xx}^i(t, x), v) \qquad (i = 1, \ldots, m) \end{aligned} \tag{1.10}$$

in D_p, if they satisfy (1.10) for $(t, x) \in D_p$.

According to the definition given by J. Szarski, a function $F^i(t, x, z, p, q, r, w)$ is called *parabolic* with respect to u in D_p if for any two real symmetric matrices $r = (r_{jk})$, $\tilde{r} = (\tilde{r}_{jk})$ and for $(t, x) \in D_p$ we have

$$\begin{aligned} r \leqslant \tilde{r}(^*) \Rightarrow F^i(t, x, u(t, x), u_t^i(t, x), u_x^i(t, x), r, u) \\ \leqslant F^i(t, x, u(t, x), u_t^i(t, x), u_x^i(t, x), \tilde{r}, u)\,. \end{aligned} \tag{1.11}$$

2. Strong inequalities.

THEOREM 2.1. *Assume that*

1° *The real functions* $F^i(t, x, z, p, q, r, w)$ $(i = 1, \ldots, m)$ *are defined for* $(t, x) \in D_p$, *where* D *is a open set satisfying Assumption* A, *for* $z \in R^m$, $p \in R$, $q \in R^n$, $r \in R^{n \times n}$, $w \in Y$ *and for every fixed index* i *the function* F^i *is weakly increasing with respect to* $z^1, \ldots, z^{i-1}, z^{i+1}, \ldots, z^m, w$ (**).

2° *The inequalities*

$$F^i(t, x, z, p, q, r, w) \geqslant F^i(t, x, z, \tilde{p}, q, r, w) \qquad (i = 1, \ldots, m)$$

are satisfied for $(t, x) \in D_p$, $z \in R^m$, $p < \tilde{p}$, $q \in R^n$, $r \in R^{n \times n}$, $w \in Y$.

(*) This inequality means that $\sum_{j,k=1}^{n} (r_{jk} - \tilde{r}_{jk}) \lambda_j \lambda_k \leqslant 0$.

(**) By the monotonicity of the function F^i with respect to w, in the sence of the relation $\overset{t}{\leqslant}$, the function F^i is an operator of Voltera type.

3° *For the given sets* $\sum_i$ *and directions* l^i $(i = 1, \dots, m)$ *satisfying Assumption* B, *for the given functions* $a^i(t, x) \geqslant 0$, *defined for* $(t, x) \in \sum_i$, *and for the given real functions* $\varphi^i(t, x, z)$, *defined for* $(t, x) \in \sum_i$, $z \in R$ *and weakly increasing with respect to* z, *the functions* u *and* v *belonging to* Y *satisfy the inequalities* (1.1), (1.2) *and the inequality*

$$u(t, x) - v(t, x) < 0 \quad \textit{for } (t, x) \in D_0 \setminus \bar{D}.$$

4° $F^i(t, x, z, p, q, r, w)$ $(i = 1, \dots, m)$ *are parabolic with respect to* u *in* D_p.

5° *The functions* u *and* v *are* $\sum$*-regular solutions of the system* (1.10) *in* D_p.

Under these assumptions we have

$$u(t, x) < v(t, x) \quad \textit{for } (t, x) \in D_p. \tag{2.1}$$

Proof. By 3° and Lemma, the set

$$\{t^* \in (t_0, t_0 + T) \colon u(t, x) < v(t, x) \quad \text{for } (t, x) \in D_p,\ t_0 < t < t^*\}$$

is non-void. Let $\tilde{t}$ be its least upper bound or $+\infty$, if it is unbounded. The assertion of Theorem 2.1 is obviously equivalent with the equality

$$\tilde{t} = t_0 + T. \tag{2.2}$$

Now, suppose the contrary holds true, i.e.

$$\tilde{t} < t_0 + T. \tag{2.3}$$

Then, by the continuity of $u(t, x) - v(t, x)$ in D_p, we would have

$$u(t, x) \leqslant v(t, x), \quad (t, x) \in D_p, \quad t_0 < t \leqslant \tilde{t}. \tag{2.4}$$

By Lemma and 3° and by the definition of $\tilde{t}$, there is an index j and a point $\tilde{x} \in S_{\tilde{t}}$ such that

$$u^j(\tilde{t}, \tilde{x}) = v^j(\tilde{t}, \tilde{x}). \tag{2.5}$$

From (2.4) and (2.5) it follows that the function $u^j(\tilde{t}, x) - v^j(\tilde{t}, x)$ attains its maximum in $S_{\tilde{t}}$ for $x = \tilde{x}$. Therefore, since $S_{\tilde{t}}$ is open and the function is of class C^2 in $S_{\tilde{t}}$, we have

$$u_x^j(\tilde{t}, \tilde{x}) = v_x^j(\tilde{t}, \tilde{x}), \tag{2.6}$$

$$u_{xx}^j(\tilde{t}, \tilde{x}) \leqslant v_{xx}^j(\tilde{t}, \tilde{x}). \tag{2.7}$$

By 5°, 4°, (2.7), 1°, (2.4), (2.5) and (2.6) we get successively

$$\begin{aligned}
0 &< F^j(\tilde{t}, \tilde{x}, u(\tilde{t}, \tilde{x}), u_t^j(\tilde{t}, \tilde{x}), u_x^j(\tilde{t}, \tilde{x}), u_{xx}^j(\tilde{t}, \tilde{x}), u) - \\
&\quad - F^j(\tilde{t}, \tilde{x}, v(\tilde{t}, \tilde{x}), v_t^j(\tilde{t}, \tilde{x}), v_x^j(\tilde{t}, \tilde{x}), v_{xx}^j(\tilde{t}, \tilde{x}), v) \\
&\leqslant F^j(\tilde{t}, \tilde{x}, u(\tilde{t}, \tilde{x}), u_t^j(\tilde{t}, \tilde{x}), u_x^j(\tilde{t}, \tilde{x}), v_{xx}^j(\tilde{t}, \tilde{x}), u) - \\
&\quad - F^j(\tilde{t}, \tilde{x}, v(\tilde{t}, \tilde{x}), v_t^j(\tilde{t}, \tilde{x}), v_x^j(\tilde{t}, \tilde{x}), v_{xx}^j(\tilde{t}, \tilde{x}), v) \\
&\leqslant F^j(\tilde{t}, x, v(\tilde{t}, \tilde{x}), u_t^j(\tilde{t}, \tilde{x}), v_x^j(\tilde{t}, \tilde{x}), v_{xx}^j(\tilde{t}, \tilde{x}), v) - \\
&\quad - F^j(\tilde{t}, \tilde{x}, v(\tilde{t}, \tilde{x}), v_t^j(\tilde{t}, \tilde{x}), v_x^j(\tilde{t}, \tilde{x}), v_{xx}^j(\tilde{t}, \tilde{x}), v).
\end{aligned}$$

Then, by the assumption 2° we have

$$v_t^j(\tilde{t}, \tilde{x}) > u_t^j(\tilde{t}, \tilde{x}) . \tag{2.8}$$

On the other hand, by (2.4) and (2.5), the function $u^j(t, \tilde{x}) - v^j(t, \tilde{x})$, defined for t in some interval $(\hat{t}, \tilde{t})$, attains its maximum at $t = \tilde{t}$. Hence we have

$$u_t^j(\tilde{t}, \tilde{x}) \geqslant v_t^j(\tilde{t}, \tilde{x}) ,$$

in contradiction with (2.8). This complets the proof of (2.2).

Remark 2.1. Theorem 2.1 is true if, instead of the parabolicity with regard to u, we assume the parabolicity with respect to v.

THEOREM 2.2. *Let assumptions* 1° *and* 2° *of Theorem* 2.1 *hold true and suppose additionally that*

6° *For the given sets* $\sum_i$ *and directions* l^i $(i = 1, \ldots, m)$ *satisfying Assumption* B, *for the given functions* $a^i(t, x) \geqslant 0$, *defined for* $(t, x) \in \sum_i$, *and for the given real functions* $\varphi^i(t, x, z)$, *defined for* $(t, x) \in \Sigma_i$, $z \in R$, *weakly increasing with respect to* z *and such that* $\varphi^i(t, x, -z) = -\varphi^i(t, x, z)$ for $(t, x) \in \sum_i$, $z \in R$, *the functions* u *and* $v \geqslant 0$ *belonging to* Y *and being* $\sum$*-regular in* D_p *satisfy the inequalities*

$$|u^i_{(+)}(t, x)| < v^i_{(+)}(t, x)$$

for $(t, x) \in (D_0 \setminus \bar{D}) \cup S_{t_0} \cup (\sum \setminus \sum_i) \cup \infty$ $(i = 1, \ldots, m)$ *and*

$$\left| \varphi^i(t, x, u^i(t, x)) - a^i(t, x) \frac{du^i(t, x)}{dl^i} \right| < \varphi^i(t, x, v^i(t, x)) - a^i(t, x) \frac{dv^i(t, x)}{dl^i}$$

for $(t, x) \in \sum_i$ $(i = 1, \ldots, m)$.

7° $F^i(t, x, z, p, q, r, w)$ $(i = 1, \ldots, m)$ *are parabolic with respect to* u *in* D_p *and*

$$F^i(t, x, -z, -p, -q, -r, -w) = -F^i(t, x, z, p, q, r, w) \tag{2.9}$$

for $(t, x) \in D_p$, $z \in R^m$, $p \in R$, $q \in R^n$, $r \in R^{n \times n}$, $w \in Y$ $(i = 1, \ldots, m)$.

8° *The functions* u *and* v *satisfy the system of equations*

$$F^i(t, x, u(t, x), u_t^i(t, x), u_x^i(t, x), u_{xx}^i(t, x), u) = 0 \qquad (i = 1, \ldots, m)$$

and the system of inequalities

$$F^i(t, x, v(t, x), v_t^i(t, x), v_x^i(t, x), v_{xx}^i(t, x), v) < 0 \qquad (i = 1, \ldots, m)$$

for $(t, x) \in D_p$.

Under these assumptions we obtain

$$|u(t, x)| < v(t, x) \quad \textit{for } (t, x) \in D_p . \tag{2.10}$$

Proof. The functions u and v satisfy all the assumptions of Theorem 2.1 and consequently we have (2.1). Then, to prove (2.10) it is sufficient to show the following inequality

$$-v(t, x) < u(t, x) \quad \text{for } (t, x) \in D_p . \tag{2.11}$$

For this purpose observe that, according to assumption 6°,

$$-v^i_{(+)}(t,x)<u^i_{(+)}(t,x)$$

for $(t,x)\in(D_0\setminus\bar{D})\cup S_{t_0}\cup(\sum\setminus\sum_i)\cup\infty$ $(i=1,\dots,m)$ and

$$\varphi^i(t,x,-v^i(t,x))-\varphi^i(t,x,u^i(t,x))<a^i(t,x)\frac{d[-v^i(t,x)-u^i(t,x)]}{dl^i}$$

for $(t,x)\in\sum_i$ $(i=1,\dots,m)$.

Moreover, assumption 8° and (2.9) imply

$$F^i(t,x,-v(t,x),-v^i_t(t,x),-v^i_x(t,x),-v^i_{xx}(t,x),-v)$$
$$>F^i(t,x,u(t,x),u^i_t(t,x),u^i_x(t,x),u^i_{xx}(t,x),u)$$

for $(t,x)\in D_p$ $(i=1,\dots,m)$.

Then (2.11) holds true, because the functions $-v$ and u satisfy all the assumptions of Theorem 2.1 (with u replaced by $-v$ and v by u).

COROLLARY. *Assume that*

1° *The real function* $F(t,x,z,p,q,r)$ *is defined for* $(t,x)\in D_p$, *where* D *is an open set satisfying Assumption* A, *for* $z\in R$, $p\in R$, $q\in R^n$, $r\in R^{n\times n}$.

2° $F(t,x,z,p,q,r)$ *is weakly decreasing with respect to* z *and* p *in* D_p.

3° *For the given set* $\Sigma_*\subset\Sigma$ *and the direction* l *satisfying Assumption* B, *for the given function* $a(t,x)\geqslant 0$, *defined for* $(t,x)\in\Sigma_*$, *and for the given real function* $\varphi(t,x,z)$, *defined for* $(t,x)\in\Sigma_*$, $z\in R$ *and strictly increasing with respect to* z, *the functions* u *and* v *belonging to* Y *satisfy the inequalities*

$$u_{(+)}(t,x)-v_{(+)}(t,x)\leqslant 0$$

for $(t,x)\in S_{t_0}\cup(\Sigma\setminus\Sigma_*)\cup\infty$ *and*

$$\varphi(t,x,u(t,x))-\varphi(t,x,v(t,x))\leqslant a(t,x)\frac{d[u(t,x)-v(t,x)]}{dl}$$

for $(t,x)\in\Sigma_*$.

4° $F(t,x,z,p,q,r)$ *is parabolic with respect to* u *in* D_p.

5° *The functions* u *and* v *are* Σ*-regular solutions of the inequality*

$$F(t,x,u(t,x),u_t(t,x),u_x(t,x),u_{xx}(t,x))$$
$$>F(t,x,v(t,x),v_t(t,x),v_x(t,x),v_{xx}(t,x)) \tag{2.12}$$

in D_p.

Under these assumptions we have

$$u(t,x)\leqslant v(t,x)\quad \textit{for } (t,x)\in D_p\,. \tag{2.13}$$

Proof. Let $\varepsilon>0$ and let

$$v^\varepsilon(t,x)=v(t,x)+\varepsilon\quad \text{for } (t,x)\in D_p\,. \tag{2.14}$$

Observe that by (2.12), (2.14) and by assumption 2° of Corollary we get.

$$\begin{aligned} &F(t, x, u(t, x), u_t(t, x), u_x(t, x), u_{xx}(t, x)) \\ &-F(t, x, v^\varepsilon(t, x), v^\varepsilon_t(t, x), v^\varepsilon_x(t, x), v^\varepsilon_{xx}(t, x)) \\ &> F(t, x, v(t, x), v_t(t, x), v_x(t, x), v_{xx}(t, x)) \\ &-F(t, x, v^\varepsilon(t, x), v^\varepsilon_t(t, x), v^\varepsilon_x(t, x), v^\varepsilon_{xx}(t, x)) \geqslant 0 \end{aligned}$$

for $(t, x) \in D_p$.

Moreover,

$$u_{(+)}(t, x) - v^\varepsilon_{(+)}(t, x) < 0 \quad \text{for } (t, x) \in S_{t_0} \cup (\Sigma \setminus \Sigma_*) \cup \infty$$

and

$$\begin{aligned} &\varphi(t, x, u(t, x)) - \varphi(t, x, v^\varepsilon(t, x)) < \varphi(t, x, u(t, x)) - \varphi(t, x, v(t, x)) \\ &\leqslant a(t, x)\frac{d[u(t, x) - v(t, x)]}{dl} = a(t, x)\frac{d[u(t, x) - v^\varepsilon(t, x)]}{dl} \quad \text{for } (t, x) \in \Sigma_*. \end{aligned}$$

Then

$$u(t, x) < v^\varepsilon(t, x) \quad \text{for } (t, x) \in D_p,$$

because the functions u and v^ε satisfy all the assumptions of Theorem 2.1. Hence (2.13) holds true.

Remark 2.2. Corollary may be formulated for a system of m nonlinear, second order degenerate functional-differential inequalities. In this case we must assume that the functions $F^i(t, x, z, p, q, r, w)$ $(i = 1, \ldots, m)$ are weakly decreasing with respect to $z = (z^1, \ldots, z^m) \in R^m$, $w \in Y$ and we must assume that for every fixed index i the function $F^i(t, x, z, p, q, r, w)$ is weakly increasing with respect to $z^1, \ldots, z^{i-1}$, $z^{i+1}, \ldots, z^m, w$. Consequently, the functions $F^i(t, x, z, p, q, r, w)$ $(i = 1, \ldots, m)$ do not depend on the arguments $z = (z^1, \ldots, z^m)$ $(m > 1)$ and w, and therefore Corollary is formulated only for one differential inequality of the type (2.12).

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