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On a Characterization of Addition in an Abelian Group

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Abstract. In this paper we solve the equation $F(x, F(y, z))+F(x, F(z, y)) = F(F(x, y),z)+F(F(y, x),z)$ for a function $F: G\times G \to G$, where G is an abelian group not possessing elements of order 2. The general solution of this equation is given in some classe of functions. The general continuous solution on the topological group R^n is also determined.

Let $(R, +)$ be the abelian group of all real numbers equipped with the usual topology. J. G. Dhombres has proved in [2] the following

Theorem. *Let F: $R\times R \to R$ be separately continuous function such that*

(i) *$F(x, y)-F(x', y)$ depends only upon $x-x'$ for all y in R.*

(ii) *There exists $(x_0, y_0) \in R\times R$ such that both $y \mapsto F(x_0, y)$ and $x \mapsto F(x, y_0)$ are not constant functions.*

(iii) $F(x, F(y, z))+F(x, F(z, y)) = F(F(x, y), z)+F(F(y, x), z)$
for all (x, y) in $R\times R$.
Then $F(x, y) = z_0+x+y$ for some $z_0 \in R$.

At the Thirteenth International Symposium of Functional Equations in May 1975 in Oberwolfach, J. G. Dhombres posed the question what remained from the theorem when R was replaced by R^n or by an abelian topological group (cf. [3], p. 230). In this paper we give the answer to this question.

Throughout the paper G will denote an abelian group (written additively). We are looking for a function F: $G\times G \to G$ satisfying the equation.

$$F(x, (F(y, z))+F(x, F(z, y)) = F(F(x, y), z)+F(F(y, x), z) \tag{1}$$

for all $x, y, z \in G$ and such that

$$F(x, y)-F(x', y) \tag{A}$$

depends only on $x-x'$ for all $y \in G$.

The main result is the following.

THEOREM 1. *Let G be a group having no elements of order 2. The general solution of equation* (1) *satisfying condition* (A) *is given by*

$$F(x,y) = \varphi(x)+\varphi(y)+c \tag{2}$$

for $x, y \in G$, where $\varphi: G \to G$ is an arbitrary solution of the Cauchy equation

$$\varphi(x+y) = \varphi(x)+\varphi(y) \tag{3}$$

for $x, y \in G$ satisfying additionaly the equation

$$\varphi^2(x) = \varphi(x) \quad \text{for } x \in G \tag{4}$$

and $c \in G$ is an arbitrary constant. ($\varphi^2(x)$ *denotes the function* $\varphi(\varphi(x))$).

Proof. Let $F: G\times G \to G$ satisfy equation (1) and condition (A). We define the functions $f: G \to G$, $g: G \to G$ as follows:

$$f(x) = F(x,o) \quad \text{for } x \in G, \tag{5}$$

$$g(x) = F(o,x) \quad \text{for } x \in G. \tag{6}$$

We obtain by (A)

$$F(x,y) = F(x,y)-F(o,y)+F(o,y) = F(x,o)-F(o,o)+F(o,y)$$
$$= f(x)+g(y)-f(o).$$

Hence we have

$$F(x,y) = f(x)+g(y)-f(o) \tag{7}$$

for $x, y \in G$ and

$$f(o) = g(o). \tag{8}$$

We shall prove that

$$f(x) = g(x) \quad \text{for } x \in G. \tag{9}$$

Putting $x = y = o$ in (1) and using denotations (5), (6) and formulas (7), (8) we obtain (after replacing z by x)

$$g^2(x)+g(f(x))+2f(o) = 2g(x)+2f^2(o). \tag{10}$$

In a similar way, putting $x = z = o$ in (1) we get

$$g^2(x)+g(f(x)) = f(g(x))+f^2(x) \tag{11}$$

and putting $y = z = o$, we have

$$2f(x)+2g^2(o) = f^2(x)+f(g(x))+2f(o). \tag{12}$$

Setting $x = 0$ in (12) and applying (8) we obtain

$$2f^2(o) = 2g^2(o).$$

Since the group G does not possess any elements of order 2 we have

(13) $$f^2(o) = g^2(o).$$

Deriving $f(g(x))$ from (11) and inserting it into (12), we get

$$2f(x)+2g^2(o) = g(f(x))+g^2(x)+2f(o).$$

The equality obtained together with (10) and (13) give

$$2f(x) = 2g(x),$$

which by the fact that G has no elements of order 2 immediately results in the relation (9). Now (7) may be rewritten as follows:

(14) $$F(x, y) = f(x)+f(y)-f(o)$$

or $x, y \in G$. Condition (A) and (14) yield a function $h: G \to G$ such that

$$f(x+y)-f(y) = h(x)$$

for $x, y \in G$, which means that the triple (f, h, f) satisfies the Pexider equation

$$f(x+y) = h(x)+f(y)$$

for $x, y \in G$. Hence, by making use of the general solution of the Pexider equation (cf. [1], p. 142), we can write the function f in the form

(15) $$f(x) = \varphi(x)+f(o)$$

for $x \in G$, where the function $\varphi: G \to G$ satisfies the Cauchy equation. The function φ satisfies also equation (4). Actually, inserting (9) into (10) we obtain

$$f^2(x) = f(x)+f^2(o)-f(o),$$

which, together with (15) and (3), implies (4). Finaly equality (2) follows immediately from (14) and (15). We have now proved that every solution of equation (1) can be written in the form (2). On the other hand it can easily be computed that a function F of form (2), with φ satisfying (3) and (4), satisfies equation (1). This completes the proof.

In virtue of Theorem 1 equation (1) may be reduced to the system of equations (3) and (4). Now we shall characterize the general solution of this system.

THEOREM 2. *A function* $\varphi: G \to G$ *satisfies equations* (3) *and* (4) *if and only if there exist subgroups* G_1, G_2 *of* G *such that*

(16) $$G = G_1 \oplus G_2$$

(i.e. *G is the direct sum of G_1 and G_2*) *and*

(17) $$\varphi(x_1+x_2) = x_1$$

or $x_1 \in G_1$, $x_2 \in G_2$.

Proof. Let $\varphi: G \to G$ by any function satisfying (3) and (4). Putting $G_1 = \varphi(G)$, $G_2 = \operatorname{Ker}\varphi$ we obtain (16) (cf. [4] Lemma 9.1). Further we have

$$\varphi(x_1+x_2) = \varphi(x_1)+\varphi(x_2) = \varphi(x_1) \tag{18}$$

for $x_1 \in G_1$, $x_2 \in G_2$.
Writing every $x_1 \in G_1$ as $x_1 = \varphi(x)$ for some $x \in G$ we get

$$\varphi(x_1) = \varphi(\varphi(x)) = \varphi(x) = x_1,$$

which, together with (18), gives (17).

Conversely, if G is of form (16) then the function defined by (17) obviously satisfies (3) and (4).

From Theorems 1 and 2 we obtain the following

COROLLARY 1. *Let G be a group having no elements of order 2 and let $F: G\times G \to G$ be a solution of equation (1) satisfying condition (A) such that for some fixed $y_0 \in G$ the function $x \mapsto F(x, y_0)$ is an injection or a surjection. Then*

$$F(x, y_0) = x+y+c \tag{19}$$

for $x, y \in G$, where $c \in G$ is a constant.

Proof. From (2) we have

$$\varphi(x) = F(x, y_0)-\varphi(y_0)-c,$$

which follows that the function φ must be an injection or a surjection. Let φ be written in the form (17). Then $G_2 = \{o\}$, hence

$$\varphi(x) = x \qquad \text{for } x \in G,$$

which, in view of Theorem 1, completes the proof.

As an immediate consequence of Theorems 1 and 2 we obtain the following

COROLLARY 2. *Let G be a group having no elements of order 2. There exists a solution F of equation (1) satisfying condition (A) which is neither of form (1) nor constant if and only if the group G can be written as the direct sum of the non-trivial subgroups.*

Now we shall consider continuous solutions of equation (1). We start with the following simple

THEOREM 3. *Let G be a topological group having no elements of order 2 and let $F: G\times G \to G$ be a solution of equation (1) satisfying condition (A) such that for some fixed $y_0 \in G$ the function $x \mapsto F(x, y_0)$ is continuous at one point. Then the functions F and φ are continuous (the function F in both variables together).*

Proof. By (2) we have

$$\varphi(x) = F(x, y_0)-F(o, y_0).$$

Hence the function φ is continuous at one point. Since φ satisfies the Cauchy equation it must be continuous everywhere, which by (2) follows that F is continuous.

Continuous solutions of equation (1) for $G = R^n$ (with the usual addition and the usual topology) are characterized in

THEOREM 4. *The general continuous solution $F: R^n \times R^n \to R^n$ of equation* (1) *satisfying condition* (A) *is given by*

$$F(x, y) = C_1 x + C_1 y + C_2 \tag{20}$$

for $x, y \in G$, where $C_2 \in R^n$ is an arbitrary constant and C_1 is an $n \times n$ matrix such that $C_1^2 = C_1$.

Proof. If $F: R^n \times R^n \to R^n$ is a continuous solution of equation (1) then, by Theorem 3, the function $\varphi: R^n \to R^n$ is also continuous, hence it may be written in the form (cf. [1], p. 215)

$$\varphi(x) = C_1 x \quad \text{for } x \in R^n,$$

where C_1 is an $n \times n$ real matrix. The equality $C_1^2 = C_1$ results directly from (4).

Conversely, it is obvious that the function F defined by (20 is continuous and is of the form (2). This completes the proof.

As an immediate consequence of Theorems 3 and 4 we obtain the following

COROLLARY 3. *Let $F: R \times R \to R$ be a solution of equation* (1) *satisfying condition* (A) *such that for some fixed $y_0 \in R$ the function $x \mapsto F(x, y_0)$ is continuous at one point. Then*

$$F(x, y) = c \tag{21}$$

for $x, y \in R$ or

$$F(x, y) = x + y + c \tag{22}$$

for $x, y \in R$, where $c \in R$ is a constant.

This corollary contains the results obtained by J. G. Dhombres in [2].

Remark. It is easy to observe that the assumption of Corollary 3, regarding the function $x \mapsto F(x, y)$ may be considerably weakend. If e.g. this function is bounded on a set of positive inner Lebesgue measure, then, in virtue of the theorem of A. Ostrowski (cf. [5]), the function φ is continuous and so is F.

In the case $G = R$ there are solutions of equation (1), which are neither of form (21) nor (22). To show this, in view of Corollary 2, it is sufficient to verify that R can be written as the direct sum of non-trivial subgroups. For this purpose, divide any base B of R regarded as a vector space over Q onto non-empty sets B_1, B_2 and put $G_1 = \mathrm{Lin}(B_1)$, $G_2 = \mathrm{Lin}(B_2)$.

To verify that the function of the form (3) satisfies equation (1), the assumption that the group G has no elements of order 2 is needless. But if we drop this assumption, then formula (2) may not give the general solution of equation (1). To show this, let us consider an

Example. Let G be any non-trivial group in which $2x = 0$ for all $x \in G$. Let $g: G \to G$, be any non-constant function such that $g^2(x) = o$ for all $x \in G$. Put

$$F(x, y) = g(y) \quad \text{for } x, y \in G.$$

We have

$$F(z, F(y, z)) + F(x, F(z, y)) = g^2(z) + g^2(y) = o$$
$$= g(z) + g(z) = F(F(x, y), z) + F(F(y, x), z).$$

But F is not of form (3). In the opposite case we should have

$$F(x, x) = \varphi(x) + \varphi(x) + C = C,$$

which is a contradiction (as g is non-constant).

References

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