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$\prec$-Prolongational Limit Sets in Generalized Pseudo-dynamical Systems

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This paper is devoted to generalized pseudo-dynamical systems (X, G, λ) (see def. 1), where (X, v) is a topological space with topology v and $(G, +, \prec)$ is an abelian ordered semigroup with a neutral element 0, which is also the minimal element.

In the system (X, G, λ) satisfying these assumptions there are defined $\prec$-prolongational limit sets corresponding to the $\prec$-limit sets introduced in [4] generalizing the notion of positive prolongational limit sets of x $(J_+(x))$ in "single valued" dynamical systems (see [1], [3]). Investigating the connections between the several variants of $\prec$-prolongational limit sets as well as their topological properties some similarities were noticed in relation both to their prototypes in the dynamical systems $J_+(x)$ and to the corresponding $\prec$-limit sets.

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Notations, definitions and examples. In the whole paper we use the following notations and definitions the majority of which are according to [2], [3] and [4].

By $(\mathscr{A}(X), 2^v)$ we denote the space of all non-empty subsets of a topological space (X, v) with the topology 2^v generated by families of the form

$$\langle V_1, \ldots, V_n\rangle := \{B \in \mathscr{A}(X)\colon B \subset \bigcup_{i=1}^{n} V_i,\ B \cap V_i \neq \emptyset \text{ for } i = 1, \ldots, n\}$$

corresponding to all finite systems of subsets of X, open in topology v. By $\mathscr{C}(X)$ we denote the family of all non-empty and compact subsets of X. ":=" denotes an equality by definition.

For arbitrary $A \in \mathscr{A}(X)$ we denote by $\mathscr{U}_A, \mathscr{V}_A, \ldots$ neighbourhoods of the set A interpreted as a point of the space $(\mathscr{A}(X), 2^v)$, i.e. the family $\mathscr{V}_A \in 2^v$, $A \in \mathscr{V}_A$. By $U_A, V_A, \ldots$ we denote open sets in the space (X, v) which contains the set A, i.e. $V_A \in v$, $A \subset V_A$.

Closure of the set $A \subset X$ in space (X, v) we denote by $\bar{A}$, and the closure of the family $\mathscr{B} \subset \mathscr{A}(X)$ interpreted as a set of space $(\mathscr{A}(X), 2^v)$ we denote by $\operatorname{cl}\mathscr{B}$.

In order to shorten the notations we introduce the map

$$\sigma\colon \mathscr{A}\big(\mathscr{A}(X)\big) \ni \mathscr{B} \to \bigcup_{B\in\mathscr{B}} B \in \mathscr{A}(X)\,.$$

The generalized pseudo-dynamical system generalizing the notion of pseudo-dynamical system has been defined (see [3]):

Definition 1. A triplet (X, G, λ) *is a generalized pseudo-dynamical system* when X is an abstract space, $(G, +)$ is an abelian semigroup with the neutral element 0, and λ is a map from $G\times X$ into $\mathscr{A}(X)$, such that for every $t, s\in G$, $x\in X$, $A\in\mathscr{A}(X)$ the conditions

$$\lambda_0(x) = \{x\} \tag{1}$$

$$\lambda_t\big(\lambda_s(A)\big)\subset\lambda_{t+s}(A) = \lambda_{s+t}(A) \tag{2}$$

are satisfied, where $\lambda_t(x) := \lambda(t, x)$ and $\lambda_t(A) := \sigma\{\lambda_t(x)\colon x\in A\}$.

Definition 2. The generalized pseudo-dynamical system (X, G, λ) is called *regular*, when the additional condition

$$\lambda_t\big(\lambda_s(A)\big) = \lambda_{t+s}(A) \tag{2'}$$

is satisfied.

In the generalized pseudo-dynamical system (X, G, λ) (see def. 1) we use the following definitions (see [4])

Definition 3. The set

$$\lambda(A) := \bigcup_{t\in G} \lambda_t(A) \tag{3}$$

is called *the zone of emission of set $A\in\mathscr{A}(X)$ in the generalized system* (X, G, λ).

The zone of emission of set $\{x\}$ in the system (X, G, λ) is called *the zone of emission of point $x\in X$ in this system.*

Definition 4. The family of sets

$$T_\lambda(A) := \{\lambda_t(A)\colon t\in G\} \tag{4}$$

is called *the trajectory of set $A\in\mathscr{A}(X)$ in the generalized system* (X, G, λ).

The trajectory of set $\{x\}$ in the system (X, G, λ) is called *the trajectory of point $x\in X$ in this system.*

Remark 1. It is clear that the zone of emission of the set $A\in\mathscr{A}(X)$ in system (X, G, λ) is a sum of sets belonging to the trajectory of set A

$$\lambda(A) = \sigma\big(T_\lambda(A)\big)\,. \tag{5}$$

Definition 5. A family $\mathscr{B}\subset\mathscr{A}(X)$ is *invariant in* (X, G, λ), when

$$T_\lambda(B)\subset\mathscr{B} \qquad \text{for } B\in\mathscr{B}\,. \tag{6}$$

DEFINITION 6. A set $Z \subset X$ is *invariant in* (X, G, λ), when

$$\lambda(z) \subset Z \quad \text{for } z \in Z . \tag{7}$$

Under the assumptions we gave at the beginning of this paper and relative to the generalized pseudo-dynamical system (X, G, λ) we use definitions of $\prec$-limit sets introduced in [4]:

DEFINITION 7. The $\prec$-limit set of the trajectory of the set $A \in \mathscr{A}(X)$ is denoted by

$$\Omega(A) := \bigcap_{t \in G} \left(\mathrm{cl}\{\lambda_s(A) \colon t \prec s,\ s \in G\}\right) . \tag{8}$$

The $\prec$-limit set of the set $A \in \mathscr{A}(X)$ is denoted by

$$L(A) := \sigma\left(\Omega(A)\right) . \tag{9}$$

The sum of $\prec$-limit sets of the points of the set $A \in \mathscr{A}(X)$ is denoted by

$$\Lambda_{\prec}(A) := \sigma\{L(x) \colon x \in A\} . \tag{10}$$

The $\prec$-limit set of zone of emission of the set $A \in \mathscr{A}(X)$ is denoted by

$$G(A) := \bigcap_{t \in G} \overline{\sigma\left(\{\lambda_s(A) \colon t \prec s, s \in G\}\right)} . \tag{11}$$

The sum of $\prec$-limit sets of zone of emission of the points of the set $A \in \mathscr{A}(X)$ is denoted by

$$\Gamma(A) := \sigma\{G(X) \colon x \in A\} . \tag{12}$$

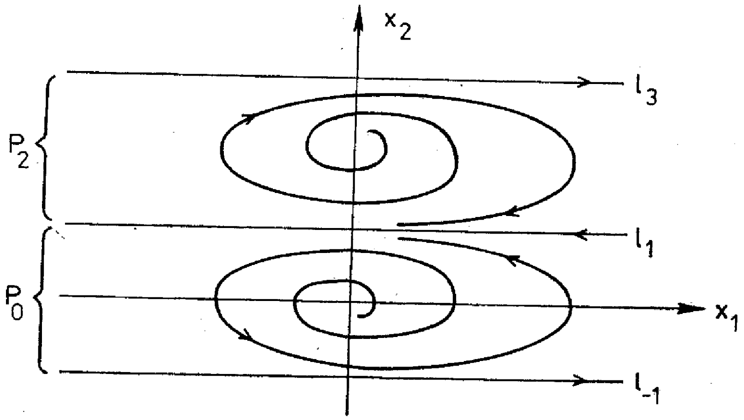
In every "single-valued" semi-system (X, G, π) in the sense of [3] we shall denote by $\Lambda(x)$ the limit set of point $x \in X$ and by $J(x)$ the prolongational limit set of point $x \in X$.

We admit that in the examples which we shall consider in this paper the space (R^2, v) is with the topology v generated by

$$K(x, r) := \{y \in R^2 \colon (x_1 - y_1)^2 + (x_2 - y_2)^2 < r^2\}$$

for $x \in R^2$, $r \in R_+$.

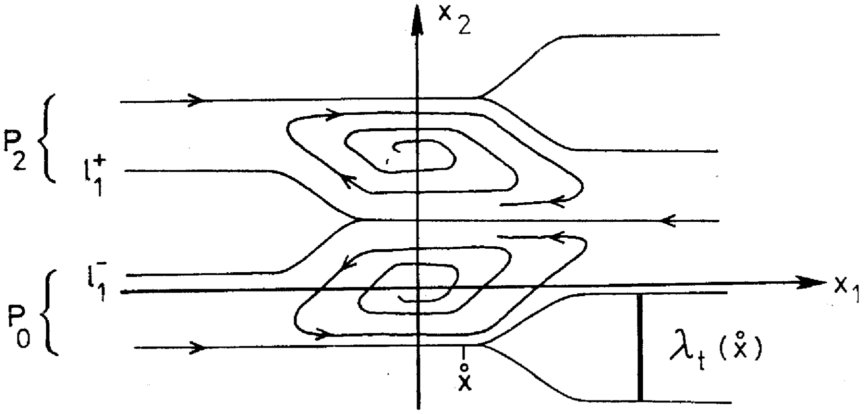
Example 1. Given a semi-system (R^2, R_*, π) (see [1]), of which trajectories are geometrically described



where $\bigcup_{i\in Z_p} P_i = R^2 \setminus \bigcup_{i\in Z_n} l_i$ for $Z_n = Z \setminus Z_p$, $Z_p = \{\dots, -2, 0, 2, \dots\}$. We define a generalized system (R^2, R_*, λ) by the following formula

$$\lambda_t(x) := \{\pi(t, x)\} \quad \text{for } t \in R_*,\ x \in R^2.$$

Note, that for every point $x \in l_i$ $(i \in Z_n)$ there is $\Omega(x) = \emptyset$, but for every point $x \in P_i$ $(i \in Z_p)$ there is $\sigma\Omega(x) = l_{i-1} \cup l_{i+1}$.



Example 2. Let us consider a generalized system (R^2, R_*, λ) as a modification of the system from Example 1; in place of a formal definition we shall give only a picture presenting the trajectories in this system. In this case $\Omega(x) = \{\{y\}\colon y \in l_{i-1}^+ \cup l_{i+1}^-\}$ for every $x \in P_i$ and $\Omega(x) = \emptyset$ for every $x \notin P_i$ $(i \in Z_p)$.

$\prec$-Prolongational limit sets. Let (X, G, λ) be a given generalized system (see def. 1), where (X, v) is a topological space with topology v and $(G, +, \prec)$ is an abelian ordered semigroup with the neutral element 0, which is aslo the minimal element.

We shall define $\prec$-prolongational limit sets in a few variants.

Definition 8. Let A be a set belonging to $\mathscr{A}(X)$.

The *$\prec$-prolongational limit set of the trajectory of A* is the family

$$J^{\Omega}(A) := \{B \in \mathscr{A}(X)\colon \forall \mathscr{V}_B \forall \mathscr{V}_A \forall t \in G\, \exists s \in G,\ t \prec s,\ \exists C \in \mathscr{V}_A \text{ such that } \lambda_s(C) \in \mathscr{V}_B\}. \tag{13}$$

The *$\prec$-prolongational limit set of the set A* is the set

$$J^{L}(A) := \sigma(J^{\Omega}(A)). \tag{14}$$

The *union of $\prec$-prolongational limit sets of the points of A* will be denoted by $J^A(A)$; thus

$$J^{A}(A) := \sigma\{J^{L}(x)\colon x \in A\}. \tag{15}$$

The $\prec$*-prolongational limit set of the zone of emission of A* is the *set*

$$J^G(A) := \{z \in X\colon\ \forall V_z \forall \mathscr{V}_A \forall t \in G\ \exists s \in G,\ t \prec s,\ \exists C \in \mathscr{V}_A \text{ such that } \lambda_s(C) \cap V_z \neq \emptyset\}\,. \tag{16}$$

The *union of* $\prec$*-prolongational limit sets of zones of emission of points x belonging to A* is denoted by $J^\Gamma(A)$; thus

$$J^\Gamma(A) := \sigma\{J^G(x)\colon\ x \in A\}\,. \tag{17}$$

We have used above (and will use in the sequel) the following short notation

$$J^K(x) := J^K(\{x\}) \qquad \text{for } K = \Omega, L, \Lambda, G, \Gamma\,. \tag{18}$$

Example 3. Let (X, G, π) be a given semi-system. Let us consider (compare Example 1) the generalized system (X, G, λ) with

$$\lambda_t(x) := \{\pi(t, x)\} \qquad \text{for } t \in G,\ x \in X\,. \tag{19}$$

It is easy to see that for this system the following connections are satisfied:

$$(z \in J(x) \Rightarrow \{z\} \in J^\Omega(x)) \qquad \text{and} \qquad J^L(x) = J(x)\,.$$

If (X, v) is a Hausdorff space, then for the generalized system already defined we have $\Omega(x) = \{\{z\}\colon\ z \in \Lambda(x)\}$.

Remark 2. For every $A \in \mathscr{A}(X)$ we have

$$K(A) \subset J^K(A) \qquad \text{for } K = \Omega, L, \Lambda_\prec, G, \Gamma\,. \tag{20}$$

Example 4. Let (X, G, π) be a (trivial) semi-system in which every point x is a critical point, i.e. $\pi(t, x) = x$ for $t \in G$, $x \in X$. We take the generalized system (X, G, λ) which is defined by (X, G, π) in the same way as in Example 3. Then the mapping λ has the form

$$\lambda\colon\ G \times X \ni (t, x) \to \{x\} \in \mathscr{A}(X)$$

and for every $A \in \mathscr{A}(X)$ we have

$$J^\Omega(A) = \Omega(A) = \{B \in \mathscr{A}(X)\colon\ A \subset B \subset \bar{A}\}, \qquad J^L(A) = L(A) = \bar{A}\,,$$
$$J^\Lambda(A) = \Lambda_\prec(A) = A\,, \qquad J^G(A) = G(A) = \bar{A} \qquad \text{and} \qquad J^\Gamma(A) = \Gamma(A) = A\,.$$

Example 5. Let us consider a generalized system (R_*^2, R_*, λ) in which the mapping λ is given by the formula

$$\lambda\colon\ (t, (\mathring{x}, \mathring{y})) \to \{(x(s, (\mathring{x}, \mathring{y})), y(s, (\mathring{x}, \mathring{y})))\colon\ s \in [0, t]\}\,,$$

where $(x(\cdot, (\mathring{x}, \mathring{y})), y(\cdot, (\mathring{x}, \mathring{y}))$ is the (unique) solution of the Cauchy initial value problem

$$\begin{aligned} \frac{dx}{dt} &= -x\,, & \frac{dy}{dt} &= y\,, \\ x(0) &= \mathring{x}\,, & y(0) &= \mathring{y}\,. \end{aligned} \tag{21}$$

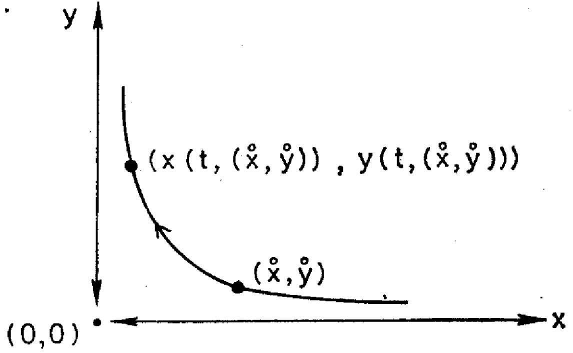
In this generalized system (R^2_*, R_*, λ) for every point $P_0 \neq (x, 0)$, $P_0 \in R^2_*$ there is

$$\Omega(P_0) = J^{\Omega}(P_0) = \{\{(x(s, P_0), y(s, P_0)):\ s \in [0, +\infty)\}\}$$

but

$$\Omega((0,0)) = \{\{(0,0)\}\} \not\supseteq J^{\Omega}((0,0)) = \{\{(0,0)\}, \{(0,y):\ y \in (0,+\infty)\},\ \{(0,y):\ y \in [0,+\infty)\}\},$$

$$L((0,0)) = \{(0,0)\} \not\supseteq J^{L}((0,0)) = \{(0,y):\ y \in [0,+\infty)\}.$$



Remark 3. If we define a semi-system (R^2_*, R_*, π) generated in the usual way by the solutions of $x' = -x$, $y' = y$ $\big(\pi(t, (\mathring{x}, \mathring{y})) = \big(x\big(t, (\mathring{x}, \mathring{y})\big), y\big(t, (\mathring{x}, \mathring{y})\big)\big)$; see (21)), then $J((0,0)) = \{(0,0)\} \not\supseteq J^L((0,0))$ where J^L is taken for the generalized system (R^2_*, R_*, λ), considered in Example 5. Notice that the equality $J^L(P) = J(P)$ does not hold true while it is satisfied e.g. in the generalized system $(R^2_*, R_*, \bar{\lambda})$, with $\bar{\lambda}$ given by the formula

$$\bar{\lambda}_t(P) := \{\pi(t, P)\} \quad \text{for } t \in R_*,\ P \in R^2_*.$$

Investigating the connections between several variants of $\prec$-prolongational limit sets in generalized systems we obtain the following results:

THEOREM 1. *For every set $A \in \mathscr{A}(X)$ the inclusions*

$$J^L(A) \subset J^G(A), \tag{22}$$

$$J^A(A) \subset J^{\Gamma}(A), \tag{23}$$

$$J^{\Gamma}(A) \subset J^G(A) \tag{24}$$

are satisfied.

Proof. In order to prove (22) we take some $z \in J^L(A)$. There is $B \in \mathscr{A}(X)$ such that $z \in B$ and $B \in J^{\Omega}(A)$, therefore for every V_z and $\mathscr{V}_B := \langle V_z, V_B \rangle$ we have by (13)

$$\forall \mathscr{V}_A \forall t \in G \exists s \in G,\ t \prec s \exists C \in \mathscr{V}_A \quad \text{such that} \quad \lambda_s(C) \in \mathscr{V}_B.$$

Hence in particular $\lambda_s(C) \cap V_z \neq \emptyset$, i.e. $z \in J^G(A)$. The inclusion (22) has been proved.

To prove (23), observe that from (22) it follows that $J^L(x) \subset J^G(x)$ for every point $x \in X$, hence for every set $A \in \mathscr{A}(X)$ is $J^A(A) \subset J^{\Gamma}(A)$.

To prove (24) let us take a point $z \in J^{\Gamma}(A)$. There is $x \in A$ such that $z \in J^{G}(x)$. For every $\mathscr{V}_A$ there are $V_i \in v$ $(i = 1, \dots, n)$ such that $\langle V_1, \dots, V_n \rangle \subset \mathscr{V}_A$, $A \in \langle V_1, \dots, V_n \rangle$. Therefore $A \subset \bigcup_{i=1}^{n} V_i$ and hence there is $i_0 \in \{1, \dots, n\}$ such that $x \in V_{i_0}$. Taking $\mathscr{V}_x := \langle V_{i_0} \rangle$ we have by (16)

$$\forall V_z \forall t \in G \exists s \in G, \quad t \prec s \exists C_x \in \mathscr{V}_x \quad \text{such that} \quad \lambda_s(C_x) \cap V_z \neq \emptyset .$$

Hence we have that for $C := A \cup C_x$ there are $C \in \mathscr{V}_A$ and $\lambda_s(C) \cap V_z \neq \emptyset$, which completes the proof.

As an immediate corollary of Definition 8 and Theorem 1 we obtain

COROLLARY 1. *For every point* $x \in X$

$$J^{L}(x) = J^{A}(x) \subset J^{G}(x) = J^{\Gamma}(x) . \tag{25}$$

Remark 4. Example 4 shows that usually for a fixed $A \in \mathscr{A}(X)$ the $\prec$-prolongational limit sets $J^{L}(A)$ and $J^{A}(A)$ as well as $J^{G}(A)$ and $J^{\Gamma}(A)$ are different sets.

Considering the space $(\mathscr{A}(X), 2^{v})$ we notice the following property.

Remark 5. For any set $C \in \mathscr{A}(X)$ every neighbourhood of C contains all the sets B satisfying condition $\operatorname{int} C \subset B \subset C$ (by $\operatorname{int} C$ we denote the interior of set C in the space (X, v)).

From this property follows

THEOREM 2. *For every set* $B \in \mathscr{A}(X)$ *the following equality*

$$\operatorname{cl}\{B\} = \{C \in \mathscr{A}(X) \colon\ B \subset C \subset \bar{B}\} . \tag{26}$$

holds true.

THEOREM 3. *For every set* $A \in \mathscr{A}(X)$

$$B \in J^{\Omega}(A) \Rightarrow \operatorname{cl}\{B\} \subset J^{\Omega}(A) .$$

Remark 6. It may happen that for some $B \in J^{\Omega}(A)$ and $C \in \mathscr{A}(X)$ there is $\operatorname{int} B \subset C \subset B$ and $C \notin J^{\Omega}(A)$.

Example 6. Taking the generalized system (X, G, λ) from Example 4 we observe that $J^{\Omega}(A) = \{A\}$ for every closed set A belonging to $\mathscr{A}(X)$.

It is known that the prototype of the $\prec$-prolongational limit sets in "single-valued" semi-systems $(J(x))$ are closed sets in space (X, v). We shall prove that in generalized systems the $\prec$-prolongational limit sets of types J^{Ω} and J^{G} are closed in the spaces $(\mathscr{A}(X), 2^{v})$ and (X, v) respectively, though the other variant $\prec$-prolongational limit sets may have not this property.

THEOREM 4. *For any* $A \in \mathscr{A}(X)$ *the family* $J^{\Omega}(A)$ *is closed in the space* $(\mathscr{A}(X), 2^{v})$.

Proof. We take an arbitrary set B belonging to $\operatorname{cl} J^{\Omega}(A)$. For every neighbourhood $\mathscr{V}_B$ there exists $B^* \in \mathscr{V}_B \cap J^{\Omega}(A)$. By conditions $B^* \in J^{\Omega}(A)$ and (13) or by definition $\mathscr{V}_{B^*} := \mathscr{V}_B$ we have

$$\forall \mathscr{V}_A \forall t \in G \exists s \in G,\ t \prec s \exists C \in \mathscr{V}_A \quad \text{such that} \quad \lambda_s(C) \in \mathscr{V}_B .$$

Because $\mathscr{V}_B$ is arbitrary we have $B \in J^{\Omega}(A)$.

THEOREM 5. *For any* $A \in \mathscr{A}(X)$ *the set* $J^{G}(A)$ *is closed in the space* (X, v).

Prood. We take an arbitrary point z which belongs to $\overline{J^{G}(A)}$. For every neighbourhood V_z there exists $z^* \in V_z \cap J^{G}(A)$. By conditions $z^* \in J^{G}(A)$ and (16) or by definition $V_{z^*} := V_z$ we have

$$\forall \mathscr{V}_A \forall t \in G \exists s \in G, \quad t \prec s \exists C \in \mathscr{V}_A \text{ such that } \lambda_s(C) \cap V_{z^*} \neq \emptyset .$$

Because V_z is arbitrary and $V_{z^*} := V_z$ the thesis is obvious.

Remark 6. $\prec$-prolongational limit sets of the type J^{A}, J^{Γ} need not be closed (see Example 4).

We prove, under certain assumptions, that $\prec$-prolongational limit sets of arbitrary type have the property of invariance.

THEOREM 6. *If* (X, G, λ) *is a regular generalized system such that for any* $p \in G$:

$$\lambda_p(\cdot)\colon\ \mathscr{A}(X) \ni B \to \lambda_p(B) \in \mathscr{A}(X) \tag{27}$$

is a continuous map in $(\mathscr{A}(X), 2^{v})$, *then for every set* $A \in \mathscr{A}(X)$ *a family* $J^{\Omega}(A)$ *is invariant in* (X, G, λ).

Proof. We take arbitrarily fixed $B \in J^{\Omega}(A)$ and $p \in G$. Since $\lambda_p(\cdot)$ is continuous (see (27)) we get

$$\forall \mathscr{V}_{\lambda_p(B)} \exists \mathscr{V}_B \quad \text{such that} \quad \forall C \in \mathscr{V}_B\ \lambda_p(C) \in \mathscr{V}_{\lambda_p(B)} .$$

Hence, by $B \in J^{\Omega}(A)$ it follows that

$$\forall \mathscr{V}_{\lambda_p(B)} \forall \mathscr{V}_A \forall t \in G \exists \tilde{s} \in G, t \prec \tilde{s} \exists C \in \mathscr{V}_A \quad \text{such that} \quad \lambda_{\tilde{s}(C)} \in \mathscr{V}_{\lambda_p(B)}$$

for $\tilde{s} = p+s$, where $s \in G$, $C \in \mathscr{V}_A$ are chosen for $\mathscr{V}_B$, which is chosen for $\mathscr{V}_{\lambda_p(B)}$. This condition shows that $\lambda_p(B) \in J^{\Omega}(A)$.

Example 7. We consider the "multi-valued" system (R^2, R_*, λ) given in Example 2. In this system for any point $\mathring{x} \in \overline{P}_i \setminus P_i$ we have

$$J^{\Omega}(\mathring{x}) = \{\{y\}\colon\ y \in l^{+}_{i-1} \cup l^{-}_{i+1}\} .$$

This equality shows that this family is not invariant. Indeed, for the point which is for example the bifurcation point, a trajectory of the set $\{y\}$ is not included in this family. It is clear that the map (27) is not continuous in $(\mathscr{A}(X), 2^{v})$ and Theorem 6 cannot be applied.

THEOREM 7. *If* (X, G, λ) *is a generalized system such that*

$$\lambda_p(\cdot)\colon\ X \ni x \to \lambda_p(x) \in \mathscr{A}(X) \quad \textit{for } p \in G \tag{28}$$

is a continuous map in (X, v), *then for every set* $A \in \mathscr{A}(X)$ *a set* $J^G(A)$ *is invariant in* (X, G, λ).

Proof. Take arbitrarily fixed $z \in J^G(A)$ and $p \in G$. Our thesis implies that for every $y \in \lambda_p(z)$ there is

$$\forall V_y \forall \mathscr{V}_A \forall t \in G \exists \tilde{s} \in G,\ t \prec \tilde{s} \exists C \in \mathscr{V}_A \quad \text{such that} \quad \lambda_{\tilde{s}}(C) \cap V_y \neq \emptyset .$$

Since $\lambda_p(\cdot)$ is continuous (see (28)) then for

$$\mathscr{V}_{\lambda_p(z)} := \langle V_y, V_{\lambda_p(z)} \rangle$$

we get

$$\forall \mathscr{V}_{\lambda_p(z)} \exists V_z \quad \text{such that} \quad \forall z^* \in V_z\ \lambda_p(z^*) \in \mathscr{V}_{\lambda_p(z)} .$$

If for every V_y we choose the neighbourhood V_z as above, then since $z \in J^G(A)$ we have

$$\forall \mathscr{V}_A \forall t \in G \exists s \in G,\ t \prec s \exists C \in \mathscr{V}_A \quad \text{such that} \quad \lambda_s(C) \cap V_z \neq \emptyset .$$

Now, for $\tilde{C} := \lambda_s(C) \cap V_z$ it holds good that $\lambda_p(\tilde{C}) \in \mathscr{V}_{\lambda_p(z)}$. In virtue of definition $\mathscr{V}_{\lambda_p(z)}$ there is

$$\emptyset \neq \lambda_p(\tilde{C}) \cap V_y \subset \lambda_p(\lambda_s(C)) \cap V_y \subset \lambda_{p+s}(C) \cap V_y ,$$

therefore for $\tilde{s} := p+s$, $t \prec s \prec \tilde{s}$ we have the proof.

The following remark is an immediate consequence of the definition of the invariance.

Remark 7. If $\mathscr{B} \subset \mathscr{A}(X)$ is an invariant family, then the set $Z := \sigma(\mathscr{B}) \in \mathscr{A}(X)$ is also invariant.

Remark 8. It is obvious that the family $\mathscr{B}$ need not be invariant even though $Z := \sigma(\mathscr{B})$ is invariant. Sets which are not invariant may belong to an invariant family.

Example 8. Let (X, G, λ) be a generalized system such that there exists a set $A \in \mathscr{A}(X)$ with the following property:

there exists $s \in G$, $s \neq 0$ such that

(i) $\forall t \in G$ is $\lambda_s(A) \neq \lambda_t(A)$

(ii) $\exists t_0 \in G$ such that $\lambda_s(A) \setminus \lambda_{t_0}(A) \neq \lambda_s(A)$.

In this system the family

$$\mathscr{B} := \{\lambda_t(A)\colon\ t \in G,\ t \neq s\} \cup \{\lambda_s(A) \setminus \lambda_{t_0}(A)\}$$