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Existence Theorems for Systems of Difference Equations

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1. This paper is concerned with the problem of existence of solutions to certain difference equations with single-valued or multi-valued right-hand sides obtained as a result of discretization of the following types of boundary value problems for ordinary differential equations:

$$y' = f(t, y), \tag{1.1}$$

$$L_1(y) = r_1, \tag{1.2}$$

$$y' \in F(t, y), \tag{1.3}$$

$$L_2(y) = r_2, \tag{1.4}$$

$$y'' = g(t, y, y'), \tag{1.5}$$

$$L_3(y) = r_3, \tag{1.6}$$

where $f\colon I \times R^d \to R^d$, $g\colon I \times R^2 \to R^2$, $L_i\colon C^1(I, R^{d_i}) \to R^{d_i}$, $I = [0, 1]$, $d, d_i \in \{1, 2, 3, \ldots\}$. The set $C^1(I, R^{d_i})$ consists of all continuously differentiable functions. The functions f, g, F, L_i also satisfy some assumptions which will be given later.

The discretization method has been used for problems (1.1), (1.2) or (1.3), (1.4) and also (1.5), (1.6) leading to a system of nonlinear equations:

$$G_1(u) = 0 \tag{1.7}$$

or

$$0 \in G(u) \tag{1.8}$$

G_1, G — depend on the right-hand sides of the these named boundary problems, the order of the differential equation and boundary conditions.

A. Lasota [4] and F. Szafraniec [6] have dealt with problems of the types (1.1), (1.2), (1.3), (1.4) while A. Lasota [3] and Z. Denkowski [2] with problems of the types (1.5), (1.6).

These authors used a method adapted from the theory of ordinary differential equations to prove the existence of solutions to (1.7) or (1.8). The purpose of this note is to present a different approach, taking advantage of sufficient conditions for the existence of the solution systems (1.7) or (1.8).

We shall prove that the conditions sufficient for the existence of the solutions given in [2], [3], [6] are in fact assumptions of Borsuk's odd mapping theorem from which the existence of solutions of considered systems will result in a simple way.

The proposed approach seems to be more natural and makes the proofs much simpler and unified. It also enables certain generalizations to be made.

In Section 2 the notations and definitions are introduced, Section 3 collects the results needed in the further parts of the paper, in Section 4 we present basic theorems on the existence of solutions of problems (1.7) or (1.8). The last Section contains the application of the basic theorems to the difference boundary value problem.

2. We shall denote by R^m the m-dimensional Euclidean space with the Euclidean norm $\|\cdot\|$. Let $c(R^m)$, $\mathrm{cf}(R^m)$, $K(R^m)$ denote respectively the set of all non-empty convex subsets of R^m, the set of all non-empty closed and convex subsets of R^m, and the set of all non-empty compact and convex subsets of R^m. We denote the ball in R^m with radius r and centre 0 by $B(r)$.

For $a, b \in R^m$ and $A, B \subset R^m$ we shall denote the Euclidean distance of a to b by $\varrho(a, b) = \|a-b\|$, and the distance of set A to set B by

$$\varrho(A, B) = \inf\{\varrho(x, y)\colon (x, y) \in A \times B\}.$$

In particular $\varrho(\{a\}, B)$, the distance of point a to set B, will be denoted by $\varrho(a, B)$. In the Cartesian product $(R^m)^n$ the norm will be introduced by

$$|u| = \sum_{i=1}^{n} \|u_i\|,$$

where $u = (u_1, \ldots, u_n) \in (R^m)^n$.

Similarly in space $(\mathrm{cf}(R^m))^n$ or $(K(R^m))^n$ we shall introduce the Hausdorff metric $d(\cdot, \cdot)$ which we shall denote as follows:

$$d(A, B) = \sum_{i=1}^{n} d(A_i, B_i),$$

where $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n) \in (\mathrm{cf}(R^m))^n$ or $(K(R^m))^n$ and $d(A_i, B_i)$ denotes the Hausdorff metric in $\mathrm{cf}(R^m)$ or $K(R^m)$ defined as follows

$$d(A_i, B_i) = \max\big(d^*(A_i, B_i), d^*(B_i, A_i)\big),$$

where

$$d^*(A_i, B_i) = \sup\{\varrho(x, B_i)\colon x \in A_i\}, \quad i = 1, \ldots, n.$$

A map $H\colon R^m \to c(R^m)$ will be called *homogeneous* if for every $\lambda \in R$, $x \in R^m$: $H(\lambda x) = \lambda H(x)$, *compact* if for any bounded subset D of R^m, the closure of the set $\bigcup_{x \in D} H(x)$ is compact in R^m, *upper semi-continuous* (u.s.c.) if its graph $\{(x, y)\colon x \in R^m, y \in H(x)\}$ is closed in $(R^m)^2$, *completely continuous* if it is compact and u.s.c., *continuous* if it is continuous in the sense of Hausdorff's metric i.e.

$$\lim_{n \to \infty} \|x_n - x_0\| = 0$$

implies that

$$\lim_{n\to\infty} d(H(x_n), H(x_0)) = 0$$

for every $x_0 \in R^m$.

The following simple lemma will be needed.

LEMMA 2.1 (see [2]). *For any integer k, a homogeneous and u.s.c. map* $H\colon R^k \to c(R^k)$ *is completely continuous.*

By Δ and ∇ we shall denote difference operators from $(R^m)^{n+1}$ to $(R^m)^{n+1}$ defined by $\Delta u = (\Delta u_0, \ldots, \Delta u_n)$, $\nabla u = (\nabla u_0, \ldots, \nabla u_n)$, where

$$\Delta u_i = \begin{cases} u_{i+1} - u_i, & i = 0, \ldots, n-1, \\ 0, & i = n; \end{cases}$$

$$\nabla u_i = \begin{cases} 0, & i = 0, \\ u_i - u_{i-1}, & i = 1, \ldots, n. \end{cases}$$

Finally, for any fixed multi-index $s = (s_1, \ldots, s_m)$, $(s_i \in \{-1, 1\})$ we define the difference operator $\Delta_s\colon (R^m)^{n+1} \to (R^m)^{n+1}$ by the formula

$$\Delta_s(u_0, \ldots, u_n) = (\Delta_s u_0, \ldots, \Delta_s u_n), \qquad (u_0, \ldots, u_n) \in (R^m)^{n+1},$$

where for $u_i = (u_i^1, \ldots, u_i^m) \in R^m$, $i = 0, \ldots, n$, we set

$$\Delta_s(u_i^1, \ldots, u_i^m) = (\Delta_{s_1} u_i^1, \ldots, \Delta_{s_m} u_i^m) \quad (\text{with } \Delta_{-1} = \nabla \text{ and } \Delta_1 = \Delta).$$

3. The following two theorems have been proved by A. Cellina [1] (p. 414) and K. Borsuk (see [5], p. 82).

THEOREM 3.1. *Let* $\Gamma\colon R^n \supset B(r) \to K(R^n)$ *be u.s.c. Then for every* $\varepsilon > 0$ *there exists a continuous mapping* $f\colon B(r) \to R^n$, *such that*

$$d(\tilde{f}, \tilde{\Gamma}) < \varepsilon,$$

where $\tilde{f}$ *and* $\tilde{\Gamma}$ *are the graphs of f and* Γ.

THEOREM 3.2. *If* $f\colon R^n \supset B(r) \to R^n$ *is a continuous mapping satisfying condition*

$$f(u) \neq \lambda f(-u)$$

for every $\|u\| = r$, $\lambda \in [0, 1]$, *then there exists an* $u^* \in B(r)$, *such that*

$$f(u^*) = 0.$$

We shall prove a lemma useful in the later part of the paper.

LEMMA 3.1. *Let* $G\colon R^n \to c(R^n)$ *be homogeneous. The following conditions are equivalent:*

$$G(u) \cap G(-u) = \emptyset \quad \text{for every } u \neq 0, \tag{3.1}$$

$$G(u) \cap \lambda G(-u) = \emptyset \quad \text{for every } u \neq 0,\ \lambda \geqslant 0, \tag{3.2}$$

$$0 \in G(u) \Rightarrow u = 0. \tag{3.3}$$

Proof. The implication (3.2) ⇒ (3.1) is obvious. For the proof of the implication (3.1) ⇒ (3.3) let us assume that there exists $\bar{u} \neq 0$, such that $0 \in G(\bar{u})$. Since G is homogeneous $0 \in G(-\bar{u})$, hence

$$G(\bar{u}) \cap G(-\bar{u}) \neq \emptyset$$

which contradicts (3.1). To show (3.3) ⇒ (3.2), suppose, on the contrary that there exists $\bar{u} \neq 0$, $\bar{\lambda} \geqslant 0$ such that

$$G(\bar{u}) \cap \bar{\lambda} G(-\bar{u}) \neq \emptyset.$$

Hence there exist v, w such that

$$v \in G(\bar{u}), \quad w \in G(-\bar{u})\colon\ v = \bar{\lambda} w.$$

Thus, from the homogeneity of G and convexity of $G(\bar{u})$ we conclude that

$$0 = \frac{v + \bar{\lambda}(-w)}{1 + \bar{\lambda}} \in G(\bar{u})$$

which contradicts (3.3).

Remark 1. Since G is homogeneous, the condition "for every $u \neq 0$" in (3.1), (3.2) may be replaced by the condition "for every $\|u\| = 1$".

Observe that for single-valued homogeneous mapping G_1 conditions (3.1)–(3.3) reduce to the following:

$$G_1(u) \neq G_1(-u) \quad \text{for every } u \neq 0, \tag{3.1$'$}$$

$$G_1(u) \neq \lambda G_1(-u) \quad \text{for every} \quad u \neq 0,\ \lambda \geqslant 0, \tag{3.2$'$}$$

$$G_1(u) = 0 \Rightarrow u = 0. \tag{3.3$'$}$$

4. THEOREM 4.1. *Let* $H\colon R^n \to c(R^n)$ *be a u.s.c. and homogeneous mapping satisfying one of the conditions* (3.1)–(3.3). *Then there exists a constant* $\beta > 0$, *such that if a map* $V\colon R^n \to c(R^n)$ *satisfies the condition*

$$\lim_{\|u\| \to \infty} \frac{d^*(V(u), H(u))}{\|u\|} \leqslant \beta \tag{4.1}$$

then

$$V(u) \cap \lambda V(-u) = \emptyset$$

for every $\|u\| = M$, $\lambda \geqslant 0$ *where* M *is sufficiently large.*

Proof. By Lemma 3.1 we may assume that (3.1) holds good. Let

$$\eta = \inf\{\varrho(H(u), H(-u)) \colon \|u\| = 1\}.$$

We shall show that $\eta > 0$. Let $\tilde{H}$ and $\tilde{H}_1$ be the graphs of the maps

$$B(1) \ni u \to H(u) \in c(R^n)$$

and

$$B(1) \ni u \to -H(u) \in c(R^n).$$

By Remark 1, (3.1) holds good if and only if $\tilde{H} \cap \tilde{H}_1 = \emptyset$. Since H is completely continuous, $\tilde{H}$ and $\tilde{H}_1$ are compact.

Hence $\varrho(\tilde{H}, \tilde{H}_1) > 0$, which implies that $\eta > 0$.

For this λ choose β and ε satisfying $0 < 2(\beta+\varepsilon) < \eta$. From (4.1) it follows that there exists an $M > 0$ such that

$$\sup\{d^*(V(u), H(u)) \colon \|u\| = M\} < (\beta+\varepsilon)M.$$

Let $\delta = \beta+\varepsilon$ and let $N \colon R^n \to c(R^n)$ be the map defined by

$$N(u) = \{w \in R^n \colon \varrho(w, H(u)) \leqslant \delta\|u\|\}.$$

It is easy to see that N is homogeneous. From the definition of δ we find that

$$N(u) \cap \lambda N(-u) = \emptyset \quad \text{for } \|u\| = M,\ \lambda \geqslant 0.$$

It is easy to see that for $\|u\| = M$, $\lambda \geqslant 0$

$$V(u) \subset N(u)$$

and

$$V(-u) \subset \lambda N(-u).$$

Indeed, if for instance $v \in V(u)$ then

$$\varrho(v, H(u)) < (\beta+\varepsilon)M = \delta\|u\|.$$

The second inclusion may be proved analogously. Hence

$$V(u) \cap \lambda V(-u) = \emptyset \quad \text{for } \|u\| = M,\ \lambda \geqslant 0$$

because sets $N(u)$, $\lambda N(-u)$ are disjoint, which completes the proof.

From Lemma 2.1, 3.1 and Theorems 3.2, 4.1 it follows immediately

COROLLARY 4.1. *Let* $G \colon R^n \to c(R^n)$ *be u.s.c. and homogeneous and satisfy one of the conditions* (3.1)–(3.3) *and let* $G_1 \colon R^n \to R^n$ *be a continuous map satisfying the condition*

$$\lim_{\|u\| \to \infty} \frac{\varrho((G_1(u), G(u))}{\|u\|} = 0. \tag{4.2}$$

Then there exists at least one solution of the equation

$$G_1(u) = 0$$

COROLLARY 4.2. *If* $G_1\colon R^n \to R^n$ *is a homogeneous and continuous map and satisfies one of the conditions* (3.1)′–(3.3)′, *then there exists a number* $\beta > 0$ *such that for every* $G\colon R^n \to c(R^n)$ *satisfying*

$$\lim_{\|u\| \to \infty} \frac{d^*(G(u), G_1(u))}{\|u\|} \leqslant \beta \tag{4.3}$$

the following condition holds good:

$$G(u) \cap \lambda G(-u) = \emptyset \tag{4.4}$$

for every $\|u\| = M$, $\lambda \geqslant 0$ *where* M *is sufficiently large.*

The proof immediately follows from Lemma 2.1, 3.1, Theorem 4.1 and from the observation that G_1 is continuous, so the mapping

$$y \to \{G_1(y)\}$$

is u.s.c.

THEOREM 4.2. *If the mapping* $G\colon B(r) \to K(R^n)$ *is completely continuous and satisfies condition* (4.4) *for every* $\|u\| = r$, $\lambda \in [0, 1]$ *then there exists* $u^* \in B(r)$ *such that*

$$0 \in G(u^*) .$$

Proof. By Theorem 3.1 there is a sequence of continuous functions $f_n\colon B(r) \to R^n$ such that

$$d(\tilde{f}_n, \tilde{G}) < \frac{1}{n}$$

here $\tilde{f}_n$ and $\tilde{G}$ are the graphs of f_n and G.

We shall prove that for sufficiently large n:

$$f_n(u) \neq \lambda f_n(-u) \quad \text{for } \|u\| = r,\ \lambda \in [0, 1] .$$

Suppose, on the contrary, that for some subsequence of sequence $[f_n\}$ denoted again by $\{f_n\}$ we can find $\|u_n\| = r$, $\lambda_n \in [0, 1]$ such that

$$f_n(u_n) = \lambda_n f_n(-u_n) .$$

From the following inequalities

$$\varrho\big((u_n, f_n(u_n)), \tilde{G}\big) < \frac{1}{n}, \quad \varrho\big((-u_n, f_n(-u_n)), \tilde{G}\big) < \frac{1}{n}$$

it results that there exist sequences $\{(\bar{u}_n, \bar{v}_n)\}$, $\{(\bar{\bar{u}}_n, \bar{\bar{v}}_n)\}$, such that

(i) $\{(\bar{u}_n, \bar{v}_n)\}, \{(\bar{\bar{u}}_n \bar{\bar{v}}_n)\} \subset \tilde{G}$,

(ii) $\varrho\big((u_n, f_n(u_n)), (\bar{u}_n, \bar{v}_n)\big) < \frac{1}{n}$,

(iii) $\varrho\big((-u_n, f_n(-u_n)), (\bar{\bar{u}}_n, \bar{\bar{v}}_n)\big) < \frac{1}{n}$.

Since G is completely continuous and $B(r)$ is compact, the sequences $\{(\bar{u}_n, \bar{v}_n)\}$, $\{(\bar{\bar{u}}_n, \bar{\bar{v}}_n)\}$ belong to a compact set. From this and from the fact that $\lambda_n \in [0, 1]$ it follows that there exist converging subsequences $\{(\bar{u}_{n_\alpha}, \bar{v}_{n_\alpha})\}$, $\{(\bar{\bar{u}}_{n_\alpha}, \bar{\bar{v}}_{n_\alpha})\}$, $\{\lambda_{n_\alpha}\}$ such that

(iv) $\lim_{n_\alpha \to \infty} \lambda_{n_\alpha} = \lambda_0 \in [0, 1]$
$f_{n_\alpha}(u_{n_\alpha}) = \lambda_{n_\alpha} f_{n_\alpha}(-u_{n_\alpha})$.

Thus by (ii) and (iii)

$$\lim_{n_\alpha \to \infty} (\bar{u}_{n_\alpha}, \bar{v}_{n_\alpha}) = \lim_{n_\alpha \to \infty} \big(u_{n_\alpha}, f_{n_\alpha}(u_{n_\alpha})\big) = (u_0, v_0),$$

$$\lim_{n_\alpha \to \infty} (\bar{\bar{u}}_{n_\alpha}, \bar{\bar{v}}_{n_\alpha}) = \lim_{n_\alpha \to \infty} \big(-u_{n_\alpha}, f_{n_\alpha}(-u_{n_\alpha})\big) = (-u_0, \bar{v}_0).$$

Moreover $\|u_0\| = r$, $v_0 \in G(u_0)$, $\bar{v}_0 \in G(-u_0)$ because $B(r)$ and $\tilde{G}$ are closed. Passing to the limit in (iv), we get

$$v_0 = \lambda_0 \bar{v}_0,$$

which is contrary to condition (4.4). Thus for sufficiently large n:

$$f_n(u) \neq \lambda f_n(-u) \quad \text{for } \|u\| = r,\ \lambda \in [0, 1].$$

By Theorem 3.2 there exists a sequence $(v_n\} \subset B(r)$ such that

$$f_n(v_n) = 0.$$

From sequence $\{v_n\}$ we can select a subsequence (marked again as $\{v_n\}$) converging to $u^* \in B(r)$. Since

$$\varrho\big((v_n, f_n(v_n)), \tilde{G}\big) = \varrho\big((v_n, 0), \tilde{G}\big) < \frac{1}{n},$$

$\tilde{G}$ is closed and it follows that

$$\varrho\big((u^*, 0), \tilde{G}\big) = 0,$$

i.e. $(u^*, 0) \in \tilde{G}$, which completes the proof.

COROLLARY 4.3. *Assume the conditions of Corollary* 4.2. *Then there exists a* $\beta > 0$, *such that for every completely continuous map* $G\colon R^n \to K(R^n)$ *satisfying condition* (4.3) *there exists a such that*

$$0 \in G(u^*).$$

The proof follows immediately from Corollary 4.2 and Theorem 4.2.

5. Applying the Theorems already given we shall show the existence of solutions for the following difference equations obtained as a result of a discrete approximation of boundary value problems (1.1), (1.2); (1.3), (1.4) or (1.5), (1.6) (see [2]).

(5.1) $\Delta u_i = f_i(u_i) \qquad i = 0, 1, \ldots, n-1,$

(5.2) $Lu = r(u) \qquad r\colon (R^m)^{n+1} \to R^m,$

$$\Delta u_i \in F_i(u_i) \qquad i = 0, 1, \dots, n-1, \tag{5.3}$$

$$Lu = 0 \tag{5.4}$$

$$\Delta u_i = A_i u_i \qquad i = 0, 1, \dots, n-1, \ A_i \text{ is an } m\times m \text{ matrix}, \tag{5.5}$$

$$\Delta u_i \in F_i(u_i) + A_i u_i \qquad i = 0, 1, \dots, n-1, \tag{5.6}$$

$$\Delta_s u_i \in F_i(u_i) \qquad i = 0, 1, \dots, n. \tag{5.7}$$

$$\Delta_s u_i = f_i(u_i) \qquad i = 0, 1, \dots, n, \tag{5.8}$$

where $F_i\colon R^m \to \mathrm{cf}(R^m)$, $L\colon (R^m)^{n+1} \to R^m$, $f_i\colon R^m \to R^m$.

Let us define mappings $F\colon (R^m)^{n+1} \to (\mathrm{cf}(R^m))^{n+1}$, $f\colon (R^m)^{n+1} \to (R^m)^{n+1}$ as a composition of mapping F_i and f_i $(i = 0, 1, \dots, n)$ respectively, i.e. $F = (F_0, F_1, \dots, F_n)$, $f = (f_0, f_1, \dots, f_n)$.

THEOREM 5.1 (see [6]). *Let mappings F and L be continuous and homogeneous and the mapping f and r be continuous. Apart from this assume that*

$$\lim_{|u|\to\infty} \frac{\varrho(f(u), F(u)) + \|r(u)\|}{|u|} = 0. \tag{5.9}$$

If problem (5.3), (5.4) *has only a trivial solution* $u = 0$, *then there exists at least one solution of problem* (5.1), (5.2).

Proof. (5.3), (5.4) may be written equivalently as a problem

$$0 \in G(u),$$

where $G\colon (R^m)^{n+1} \to (\mathrm{cf}(R^m))^{n+1}$, $u = (u_0, u_1, \dots, u_n)$

$$G(u) = \begin{bmatrix} \bar{F}(u) - Au \\ \{Lu\} \end{bmatrix}, \quad \bar{F} = (F_0, F_1, \dots, F_{n-1}), \quad A = \begin{bmatrix} -\mathrm{I}, & \mathrm{I}, & 0,\ 0,\ \dots,\ 0 \\ 0, & -\mathrm{I}, & \mathrm{I},\ 0,\ \dots,\ 0 \\ 0, & 0, & -\mathrm{I},\ \mathrm{I},\ \dots,\ 0 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ 0, & \dots\dots, & -\mathrm{I},\ \mathrm{I} \end{bmatrix}$$

is the $mn \times m(n+1)$ block-matrix, and problem (5.1), (5.2) as

$$G_1(u) = 0$$

where $G_1\colon (R^m)^{n+1} \to (R^m)^{n+1}$,

$$G_1(u) = \begin{bmatrix} \bar{f}(u) - Au \\ Lu - r(u) \end{bmatrix}, \quad \bar{f} = (f_0, f_1, \dots, f_{n-1}).$$

Observe that G as a continuous mapping is u.s.c. G is also homogeneous. The assumption that (5.3), (5.4) has only a trivial solution implies that

$$0 \in G(u) \Rightarrow u = 0.$$

From (5.9) it follows that

$$\lim_{|u|\to\infty} \frac{\varrho(G_1(u), G(u))}{|u|} = \lim_{|u|\to\infty} \frac{\varrho(\bar{f}(u), \bar{F}(u)) + ||r(u)||}{|u|} \leqslant$$

$$\leqslant \lim_{|u|\to\infty} \frac{\varrho(f(u), F(u)) + ||r(u)||}{|u|} = 0 .$$

By Corollary 4.1 we obtain the existence of a solution of the equation $G_1(u) = 0$.

THEOREM 5.2. (comp [6]). *Let mapping* $F\colon (R^m)^{n+1} \to (\mathrm{cf}(R^m))^{n+1}$ *be u.s.c., and let* r *be continuous and* $L\colon (R^m)^{n+1} \to R^m$ *linear. Also, let the following conditions*

$$d^*(F_i(v), \{0\}) \leqslant \alpha_i + \beta_i ||v|| \qquad i = 0, 1, \ldots, n-1 . \tag{5.10}$$

$$\lim_{|u|\to\infty} \frac{||r(u)||}{|u|} = 0 \tag{5.11}$$

be satisfied.

If problem (5.4), (5.5) *has only a solution* $u = 0$, *then there exists a number* $\beta > 0$ *such that for every* F *u.s.c. and* r *satisfying* (5.10), (5.11) *with* $\sum_{i=0}^{n-1} \beta_i < \beta$ *problem* (5.2), (5.6) *has at least one solution.*

Proof. Setting

$$G(u) = \begin{bmatrix} \bar{F}(u) - (\bar{A} - A)u \\ \{Lu - r(u)\} \end{bmatrix}, \qquad G_1(u) = \begin{bmatrix} (\bar{A} - A)u \\ Lu \end{bmatrix},$$

where $\bar{F}, A$ such as in the proof of Theorem 5.1

$$\bar{A} = \begin{bmatrix} A_0, & 0, & 0, & \ldots, & 0 \\ 0, & A_1, & 0, & \ldots, & 0 \\ 0, & 0, & A_2, & \ldots, & 0 \\ 0, & \ldots & \ldots & \ldots, & 0 \\ 0, & \ldots & \ldots & \ldots, & 0 \\ 0, & \ldots & \ldots & A_{n-1}, & 0 \end{bmatrix} \quad \text{is an } mn \times m(n+1) \text{ block-matrix}$$

we find that

1°. G_1 as a linear mapping is continuous and homogeneous,

2°. G is completely continuous, on the basis that F is u.s.c., L is linear, r is continuous and of conditions (5.10), (5.11),

3°. the assumption that problem (5.4), (5.5) has only a trivial solution, means that implication $0 = G_1(u) \Rightarrow u = 0$ is true,