

A note on degenerate pseudo-Riemannian manifolds

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1. Introduction. In this paper we shall determine the first structure tensor of pseudo-Riemannian structure with degenerate metric and prove the necessary and sufficient condition for the existence of torsion-free $O(f)$ -connection.

The deformation tensor for torsion-free $O(f)$ -connection is given. Let M be a real Hausdorff paracompact smooth manifold, $\dim(M) = n$ and denote by g a positive semi-definite covariant symmetric tensor field of degree 2 on M .

DEFINITION 1.1. The Riemannian manifold (M, g) is called an r -time degenerate if $\text{rank}(g) = n - r$, where $1 \leq r \leq n - 1$.

DEFINITION 1.2. If there is an atlas of local maps in which the tensor field g has the form

$$g = \left[\begin{array}{c|c} g & 0 \\ \hline 0 & 0 \end{array} \right] \quad \text{and } \det(g) \neq 0, \quad \text{where } g = (g_{ab}) \quad a, b = 1, \dots, n-r$$

then (M, g) is called *reducible*.

DEFINITION 1.3. If there is an atlas of local maps in which (M, g) is reducible and $\partial_A g = 0$ for $A = n - r + 1, \dots, n$ then (M, g) is called *absolutely reducible*.

THEOREM 1.1. (O. Vogel). *Let (M, g) be an r -time degenerate Riemannian manifold. (M, g) admits a metric torsionless linear connection if and only if it is absolutely reducible.*

We shall prove an equivalent version of Vogel's theorem in the G -structure language and discuss the variety of the connection in questions. For proof of Theorem 1.1. see [1].

2. The degenerate pseudo-Riemannian structure and flatness problem. Let f be an r -time degenerate symmetric bilinear form on $V \times V$, where we denote by V the real vector space, $\dim(V) = n$, the signature of f is defined by $(+, \dots, +, -, \dots, -, \underbrace{0, \dots, 0}_{r\text{-times}})$.

The degenerate pseudo-orthogonal group of f is

$$O(f) := \{a \in GL(V) \mid \forall x, y \in V f(ax, ay) = f(x, y)\}.$$

The Lie algebra of $O(f)$ is

$$o(f) := \{A \in gl(V) \mid \forall x, y \in V f(Ax, y) = -f(x, Ay)\}.$$

Let $L(M, p, GL(n, R))$ be the principal bundle of linear frames over M . We denote the reduced bundle of $L(M, p, GL(n, R))$ to the Lie subgroup $O(f)$ by $O(M, p, O(f))$ where $p := p|_O$ is the canonical projection.

DEFINITION 2.1. The $O(M, p, O(f))$ -structure is called a *degenerate pseudo-Riemannian structure*.

Let g be a Lie subalgebra of $gl(V)$.

DEFINITION 2.2. The *first prolongation* $g^{(1)}$ of g is defined by

$$g^{(1)} := g \otimes V^* \cap V \otimes S^2(V^*),$$

where $S^2(V^*)$ denotes the spaces of the symmetric tensor of degree 2 over V . In general

$$g^{(k)} := g \otimes \underbrace{V^* \otimes V^* \otimes \dots \otimes V^*}_{k\text{-times}} \cap V \otimes S^{k+1}(V^*).$$

We say that g is of *finite type* if $g^{(k)} = 0$ for some k . If $g^{(k)} \neq 0$ for all k then g is said to be of *infinite type*.

LEMMA 2.1. Let $\ker(f) := \{x \in V \mid \forall y \in V f(x, y) = 0\}$ and V be a complement to $\ker(f)$ in V . Then

- (1) $o(f) = o(f) \oplus \ker(f) \otimes V^*$, where $f := f|_V$,
- (2) $o(f)^{(1)} = \ker(f) \otimes S^2(V^*)$ and the Lie algebra $o(f)$ is of infinite type.

Proof. ad(1). Corresponding to the direct sum decomposition $V = V \oplus \ker(f)$, there is decomposition of $V \otimes V^*$:

$$V \otimes V^* = V \otimes V^* \oplus V \otimes \ker(f)^* \oplus \ker(f) \otimes V^* \oplus \ker(f) \otimes \ker(f)^*$$

but for all $A \in o(f)$, $x \in V$, $k \in \ker(f)$ since

$$f(Ak, x) = -f(k, Ax) = 0 \text{ it follows that } A(\ker(f)) \subset \ker(f).$$

Hence in the corresponding decomposition of any $A \in o(f)$ the $V \otimes \ker(f)^*$ component is zero. Let A denote the restriction of A to V and A_1 its $V \otimes V^*$ component, then for all $x = x+k$, $y = y+l \in V \oplus \ker(f) = V$, we have

$$\begin{aligned} f(Ax, y) + f(x, Ay) &= f(Ax, y) + f(x, Ay) = 0 \\ &= f(A_1x, y) + f(x, A_1y) \end{aligned}$$

hence $A_1 \in o(f)$ and $o(f) \subset o(f) \oplus \ker(f) \otimes V^*$.

