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On Convergence of Iterates of the Frobenius-Perron Operator

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Abstract. It is shown that the sequence $P_\tau^n f$ is convergent in norm topology for any $f \in L^1$ and is uniformly convergent for any function with the bounded variation over $[0, 1]$ where P_τ is the Frobenius-Perron operator for a certain generalization of Rényi's transformations.

1. Introduction. In the paper [6] A. Lasota and J. A. Yorke have shown that if $\tau: [0, 1] \to [0, 1]$ is a piecewise C^2 transformation and $\inf|\tau'| > 1$, then for any $f \in L^1$ the sequence

$$\frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f,$$

where P_τ denotes the Frobenius—Perron operator corresponding to τ is convergent in norm to a function $f^* \in L^1$ which is the density of an invariant measure under τ. K. Krzyżewski has stated [13] that for any continuous function f, $P_\tau^n f$ is uniformly convergent whenever $\tau: M \to M$ is a C^2 expanding transformation from a compact connected differentiable manifold M into itself. The main purpose of the present note is to show that if τ is a generalization of Renyi's transformation then, the sequence of iterates $P_\tau^n f$ is uniformly convergent for any f with the bounded variation over $[0, 1]$ and consequently, strongly convergent for any $f \in L^1$. The limit of $P_\tau^n f$ is the density of a measure invariant under τ.

To prove this theorem we first show that $P_\tau^n f$ is quasi-equicontinuous for f with the bounded variation and weakly convergent in L^2 for any $f \in L^2$. From this and a generalization of the Arzela theorem we obtain our results.

In Section 2 we recall some basic definitions and state the main theorem. In Section 3 we prove some necessary lemmas and the theorem.

2. Convergence theorem. Denote by $L^p(X, \Sigma, \mu)$ the space of all integrable functions $f: X \to R$ with p-th power. The Lebesgue measure on $[0, 1]$ will be denoted by m.

We say that a measurable transformation $\tau: [0, 1] \to [0, 1]$ is nonsingular if $m(\tau^{-1}(A)) = 0$ whenever $m(A) = 0$ for any measurable set A.

For nonsingular τ: $[0, 1] \to [0, 1]$ we define the Frobenius—Perron operator P_τ: $L^1([0, 1], \Sigma, m) \to L^1([0, 1], \Sigma, m)$ by the formula

$$\int_A P fdm = \int_{\tau^{-1}(A)} fdm$$

which is valid for each measurable set $A \subset [0, 1]$. It is well known that the operator P_τ is linear and satisfies the following conditions

(a) P_τ is positive: $f \geqslant 0 \Rightarrow P_\tau f \geqslant 0$,

(b) P_τ preserves integrals

$$\int_0^1 P_\tau \; fdm = \int_0^1 fdm \,, \quad f \in L^1,$$

(c) $P_{\tau^k} = P_\tau^k$ (τ^k denotes the n-th iterate of τ),

(d) $P_\tau f = f$ if and only if the measure $d\mu = fdm$ is invariant under τ i.e., $\mu(\tau^{-1}(A)) = \mu(A)$ for each measurable A.

We shall not make a distinction between functions f: $[0, 1] \to R$ defined on $[0, 1]$ and functions f: $[0, 1] \to R$ taken as elements of space L^p for $p \geqslant 1$. This difference will become clear in the context.

A transformation τ: $[0, 1] \to R$ will be called piecewise C^2, if there exists a partition $0 = a_0 < a_1 < \ldots < a_p = 1$ of the unit interval such that for each integer i ($i = 1, 2, \ldots, p$) the restriction τ_i of τ to the open interval (a_{i-1}, a_i) is a C^2 function which can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^2 function. τ need not be continuous at the points a_i.

THEOREM 1. *If the transformation* τ: $[0, 1] \to [0, 1]$ *is a piecewise* C^2 *function for which there exists a partition* $0 = a_0 < a_1 < \ldots < a_p = 1$ *of* $[0, 1]$ *such that for each integer* i ($i = 1, 2, \ldots, p$) *the restriction* τ_i *of* τ *to the open interval* (a_{i-1}, a_i) *can be extended to the closed interval* $[a_{i-1}, a_i]$ *as a* C^2 *an bijective map of interval* $[a_{i-1}, a_i]$ *onto* $[0, 1]$ *and* $s = \inf|\tau'(x)| > 1$, *then there exists exactly one continuous function* $u \in L^1$ *such that*

(i) *for any* $f \geqslant 0, f \in L^1$ *with the bounded variation over* $[0, 1]$ *the sequence of functions* $P_\tau^n f$ *is uniformly convergent to the function* $||f||u$,

(ii) *for any* $f \geqslant 0, f \in L^1$ *the sequence* $P_\tau^n f$ *is convergent in norm to the function* $||f||u$,

(iii) *measure* $d\mu = u||f||dm$ *is invariant under* τ.

3. Auxiliary lemmas and proof of the convergence theorem.

In the proof Theorem 1 we shall use the following lemmas and theorems.

By a direct modification of the proof of the inequality $m(\tau^{-n}(A)) \geqslant Lm(A)$ (see [3]) we may obtain the proof of the following lemma.

LEMMA 1. *If the transformation* τ: $[0, 1] \to [0, 1]$ *satisfies the assumptions of Theorem* 1, *then there exists a constant L such that for any measurable set* $A \subset [0, 1]$ *and* $n = 1, 2, \ldots$

$$m(\tau^{-n}(A)) \leqslant Lm(A) \,.$$

A sequence of functions $\{h_n\}_{n=1}^{\infty}$, h_n: $[0, 1] \to R$, is said to be quasi-equicontinuous on $[0, 1]$ if for every $\varepsilon > 0$ there exists n_0 and $\delta > 0$ such that

$$|h_n(x) - h_n(y)| < \varepsilon$$

whenever $|x - y| < \delta$, $x, y \in [0, 1]$ and $n > n_0$.

For the proof of Theorem 1 we shall need the following generalization of the Arzela theorem.

THEOREM (Arzela). *If a sequence* $\{h_n\}_{n=1}^{\infty}$, h_n: $[0, 1] \to R$, *is uniformly bounded on* $[0, 1]$ *and quasi-equicontinuous on* $[0, 1]$, *then*

(i) $\{h_n\}_{n=1}^{\infty}$ *contains an uniformly convergent subsequence* $\{h_{n_j}\}_{j=1}^{\infty}$,

(ii) $\lim h_{n_j}$ *is a continuous function.*

The proof of this theorem is identical with the proof of the well-known Arzela theorem.

Let $A = \bigcup_{i=1}^{m} [c_i, d_i] \subset [0, 1]$ be such that $[c_j, d_j] \cap [c_i, d_i] = \emptyset$ for $i \neq j$ and let f: $[0, 1] \to R$. We define the variation of f over the set A by the formula

$$\bigvee_A f = \sum_{i=1}^{m} \bigvee_{c_i}^{d_i} f.$$

LEMMA 2. *If the transformation* τ: $[0, 1] \to [0, 1]$ *satisfies the assumptions of Theorem* 1, *then for any function* $f \geqslant 0$ *of bounded variation over* $[0, 1]$ *the sequence* $P^n f$ *is uniformly bounded and quasi-equicontinuous on* $[0, 1]$.

Proof. Set $\varphi_i(x) = \tau_i^{-1}(x)$ where τ_i: $[a_{i-1}, a_i] \to [0, 1]$ is a C^2 function such that $\tau_i|(a_{i-1}, a_i) = \tau|(a_{i-1}, a_i)$. A simple computation shows that the Frobenius-Perron operator corresponding to τ may be written in the form

$$P_\tau f(x) = \sum_{i=1}^{p} f(\varphi_i(x))|\varphi_i'(x)| .$$

By its very definition the operator P_τ is a mapping from L^1 into L^1, but the last formula enables us to consider P_τ as a map from the space of functions defined on $[0, 1]$ into itself.

By the same method as in [6] it is easy to verify that there exists K such that for every function $f \geqslant 0$ with the bounded variation over $[0, 1]$ we have

$$\bigvee_0^1 P_\tau^n f \leqslant ||f|| K \frac{1}{1 - s^{-1}} + \bigvee_0^1 f \leqslant M , \quad n = 1, 2, \ldots \tag{1}$$

Therefore, the sequence $P_\tau^n f$ is uniformly bounded on $[0, 1]$.

Let $A = \bigcup_{j=1}^{m} A_j$, where $A_j = [c_j, d_j] \subset [0, 1]$ and $A_j \cap A_i = \emptyset$ for $i \neq j$. For $f \geqslant 0$ with the bounded variation over $[0, 1]$ we have

$$\begin{aligned}\bigvee_A P_\tau f &= \sum_{j=1}^{m} \bigvee_{A_j} P_\tau f = \sum_{j=1}^{m} \bigvee_{A_j} \sum_{i=1}^{p} f(\varphi_i)|\varphi_i'| \\ &\leqslant \sum_{j=1}^{m} \sum_{i=1}^{p} \bigvee_{A_j} f(\varphi_i)|\varphi_i'| \leqslant \sum_{j=1}^{m} \sum_{i=1}^{p} \int_{A_j} |d(f(\varphi_i)|\varphi_i'|)| \\ &\leqslant \sum_{j=1}^{m} \sum_{i=1}^{p} \int_{A_j} f(\varphi_i)|\varphi_i''|\,dm + \int_{A_j} |\varphi_i'||df(\varphi_i)| \\ &\leqslant \sum_{j=1}^{m} \sum_{i=1}^{p} K \int_{\tau_i^{-1}(A_j)} f\,dm + s^{-1} \int_{\tau_i^{-1}(A_j)} |df| \\ &= K \int_{\tau^{-1}(A)} f\,dm + s^{-1} \bigvee_{\tau^{-1}(A)} f,\end{aligned}$$

where

$$K = \frac{\max_i \sup\{|\varphi_i''(x)|:\ x \in [0, 1]\}}{\min_i \inf\{|\varphi_i'(x)|:\ x \in [0, 1]\}}.$$

The last inequality gives us

$$\bigvee_A P_\tau^n f \leqslant K \int_{\tau^{-1}(A)} P_\tau^{n-1} f\,dm + Ks^{-1} \int_{\tau^{-2}(A)} P^{n-2} f\,dm + \ldots + $$
$$+ Ks^{-n+1} \int_{\tau^{-n}(A)} f\,dm + s^{-n} \bigvee_{\tau^{-n}(A)} f.$$

From (1), Lemma 1 and the last inequality we obtain

$$\bigvee_A P_\tau^n f \leqslant K(M + \|f\|) Lm(A) \frac{1}{1 - s^{-1}} + s^{-n} \bigvee_{\tau^{-n}(A)} f. \tag{2}$$

Since for any $h\colon R \to R$ $|h(x_1) - h(x_2)| \leqslant \bigvee_{x_1}^{x_2} h$, from (2) for $A = [x_1, x_2]$ we obtain that the sequence $P_\tau^n f$ is quasi-equi-continuous. This completes the proof.

Using the Arzela theorem it is easy to verify that the following is valid

LEMMA 3. *If the sequence of functions* $\{h_n\}_{n=1}^{\infty}$, $h_n\colon [0, 1] \to R$, *is quasi-equicontinuous on* $[0, 1]$ *and is weakly convergent in* L^2 *to a function* h, *then* $\{h_n\}$ *is uniformly convergent to* h *and* h *is a continuous function.*

LEMMA 4. *If the transformation* $\tau\colon [0, 1] \to [0, 1]$ *satisfies the assumptions of Theorem* 1, *then*

(i) *there exists exactly one function* $u^* \in L^1$ *such that* $\|u^*\| = 1$ *and the measure* $d\mu = u^* dm$ *is invariant under* τ,

(ii) $u^* \geqslant 0$,

(iii) u^* *is continuous on* $[0, 1]$,

(iv) *there exists constant* $c>0$ *such that*

$$\frac{1}{c} \leqslant u^*(x) \leqslant c .$$

Proof. Applying the Yorke–Li theorem [11] it is easy to verify (i) and (ii). Since the sequence

$$\frac{1}{n} \sum_{k=1}^{n-1} P_\tau^{k_1}$$

converges in norm to u^* (see [6]) from Lemmas 2 and 3 we obtain (iii), so that, it only remains to prove (iv). Suppose that there exists $x_0 \in [0, 1]$ such that $u^*(x_0) = 0$. Since $u^*(x) \geqslant 0$ and

$$u^*(x) = \sum_{i=1}^{p} u^*(\varphi_i(x))|\varphi_i'(x)|$$

therefore, $u^*(x) = 0$ for $x \in \tau^{-1}(x_0)$ and consequently $u^*(x) = 0$ on the set

$$B = \bigcup_{n=0}^{\infty} \tau^{-n}(x_0) .$$

Since the set B is dense in $[0, 1]$ and $u^*(x)$ is continuous, therefore $u^*(x) \equiv 0$ on $[0, 1]$. This is impossible, because $\|u^*\| = 1$. From this and the continuity of u^* we obtain (iv). This completes the proof.

A measure-preserving transformation τ on the probability space (X, Σ, μ) is said to be strong-mixing if for any $A, B \in \Sigma$

$$\lim_{n\to\infty} \mu(\tau^{-n}(A) \cap B) = \mu(A)\mu(B) .$$

Rohlin has proved in [9]

LEMMA 5. *If the transformation* $\tau\colon [0, 1] \to [0, 1]$ *satisfies the assumptions of Theorem* 1, *then* τ *is strong-mixing.*

LEMMA 6. *Measure-preserving transformation* $\tau\colon X \to X$ *is strong-mixing if and only if for any* $f, g \in L^2(X, \Sigma, \mu)$

$$\lim_{n\to\infty} \int_X f(\tau^n(x))g\,d\mu = \int_X f\,d\mu \int_X g\,d\mu .$$

The proof of this lemma is given in [12].

The following lemma has been proved by A. Lasota.

LEMMA 7. *If the transformation* $\tau\colon [0, 1] \to [0, 1]$ *satisfies the assumptions of Theorem* 1, *then for any* $h \in L^2$ $([0, 1], \Sigma, m)$ *the sequence* $P_\tau^n h$ *is weakly convergent in* $L^2([0, 1], \Sigma, m)$

to the function

$$h^* = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} P_\tau^k h .$$

Proof. From Lemmas 5 and 6 it follows that

$$\lim_{n\to\infty} \int_0^1 h(x) g(\tau^n(x)) d\mu = \int_0^1 h d\mu \int_0^1 g d\mu$$

for $h, g \in L^2([0, 1], \Sigma, \mu)$, where μ is the invariant measure under τ. Since

$$\int_0^1 h(x) g(\tau^n(x)) d\mu = \int_0^1 (P_\tau^n h) g d\mu$$

therefore

$$\lim_{n\to\infty} \int_0^1 (P_\tau^n h) g d\mu = \int_0^1 h d\mu \int_0^1 g d\mu .$$

By applying Lemma 4 we have

$$\int_0^1 (P_\tau^n h) g dm = \int_0^1 P_\tau^n h \frac{g}{u^*} u^* dm = \int_0^1 P_\tau^n h \frac{g}{u^*} d\mu .$$

Therefore, the sequence $\int (P_\tau^n h) g dm$ is convergent for every $h, g \in L^2([0, 1], \Sigma, m)$ to $\int_0^1 hu^* dm \int_0^1 g dm$.

From this, by the reflexivity of $L^2([0, 1], \Sigma, m)$ for every $h \in L^2([0, 1], \Sigma, m)$ there exists $\bar{h} \in L^2([0, 1], \Sigma, m)$ such that the sequence $P_\tau^n h$ is weakly convergent in $L^2([0, 1], \Sigma, m)$ to $\bar{h}$. It is easy to see that $P_\tau \bar{h} = \bar{h}$. By the Lasota—Yorke theorem [6] and Lemma 4 $h^* = \bar{h}$. This completes the proof.

Proof of Theorem 1. By Lemmas 2, 3, 4 and 7 we have (i). Let $f \in L^1$, $f \geqslant 0$. Since the set of functions with the bounded variation is dense in L^1 therefore for $\varepsilon > 0$ there exists $\bar{f} \geqslant 0$ with the bounded variation such that

(3) $$\|\bar{f} - f\| \leqslant \varepsilon .$$

Since the operator

$$Q = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k$$

is continuous (see [6]) setting

$$f^* = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f \quad \text{and} \quad \bar{f}^* = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k \bar{f}$$

from (3) and (i) we have

$$\|P_\tau^n f - f^*\| \leqslant \|P_\tau^n f - P_\tau^n \bar{f}\| + \|P_\tau^n \bar{f} - \bar{f}^*\| + \|\bar{f}^* - f^*\| \leqslant 3\varepsilon$$

for sufficiently large n. This proves (ii). It is obvious that $d\mu_f = \|f^*\| u^* dm$ is invariant under τ. This completes the proof of the theorem.

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