

On the Convergence to Zero of Oscillating Solutions of N -th Order Partial Differential Equations

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In this paper we shall prove two theorems on the convergence to zero of some special solutions of partial differential equations. The conception of these solutions is a generalization of a conception of quickly oscillating solutions of ordinary equations in the papers [1] and [2].

Let n be an integer ≥ 1 . The small Greek letters α and β denote n -indices: $\alpha = (\alpha_1, \dots, \alpha_n)$, α_i are integers ≥ 0 . We put

$$|\alpha| = \sum_{i=1}^n \alpha_i.$$

We introduce the following notation:

$$\alpha \geq \beta \Leftrightarrow \alpha_i \geq \beta_i \quad \text{for } i = 1, \dots, n,$$

$$\alpha > \beta \Leftrightarrow \alpha_i > \beta_i \quad \text{for } i = 1, \dots, n,$$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

(if $\alpha = 0, \dots, 0$, then $D^\alpha \varphi = \varphi$ for any function φ).

We shall consider the following equation of the N -th order:

$$(1) \quad D^\alpha \varphi = f(x, (D^\beta \varphi)_{\beta \leq \alpha, \beta \neq \alpha})$$

when $x = (x_1, \dots, x_n)$, $|\alpha| = N$

and $f: \mathbf{R}^q \rightarrow \mathbf{R}$ is a continuous function,

$$f = f(x, (u_\beta)_{\beta \leq \alpha, \beta \neq \alpha}), \quad q = n + \prod_{i=1}^n (\alpha_i + 1) - 1.$$

Let $\{r_\nu\}_{\nu=1}^\infty$ be the sequence of positive real numbers, such that $r_\nu \rightarrow \infty$ when $\nu \rightarrow \infty$
 $r_{\nu+1} > r_\nu$

$$\sqrt{r_{\nu+1}^2 - r_\nu^2} \rightarrow 0 \quad \text{when } \nu \rightarrow \infty.$$

For $i = 1, \dots, n$ let H_v^i be the set contained in the ring $\{x \in \mathbf{R}^n: r_v \leq |x| \leq r_{v+1}\}$. ($|\cdot|$ is the Euclidean norm). We assume that H_v^i is the boundary of connected set, which contains a ball with the centre in zero and the radius r_v .

Let k be an integer, $1 \leq k \leq n$. We say that the solution of the equation (1) is $[k, \{r_v\}]$ pseudo-oscillating, if

(a) $D^\gamma \varphi$ is defined and continuous in $[0, \infty)^n$, for $0 \leq \gamma \leq \alpha$,

(b) $\sup\{|D^\gamma \varphi(x)|: x \in H_v^{j+1}\} \rightarrow 0, v \rightarrow \infty$,

where $\gamma = 0$ for $j = 0$ and $\gamma = (\alpha_1, \dots, \alpha_j, 0, \dots, 0)$ for $j = 1, \dots, k-1$,

(c) $\exists M > 0: \sup\{|D^\gamma \varphi(x)|: x \in H_v^{j+1}\} < M$

for $j = k, \dots, n-1$ and γ as above.

Remark 1. For $n = 2$ and the hyperbolical equation the problem of the existence of solutions which fulfil (a) was considered in [4]. In this case the conception of a $[1, \{r_v\}]$ pseudo-oscillating solution generalizes the conception of an x -regular $\{r_v\}$ oscillating solution in [3].

Let $A_k = A_k(\alpha)$ be a set of n -indices β such that

$$\beta_1 = \alpha_1, \dots, \beta_{s-1} = \alpha_{s-1}, \quad 0 \leq \beta_s < \alpha_s, \quad \beta_{s+1} = 0, \dots, \beta_n = 0$$

for some $s \leq k$.

THEOREM 1. Let k be an integer, $1 \leq k \leq n$, and $\alpha = (\alpha_1, \dots, \alpha_n)$ be an n -index such that $\alpha_k \neq 0$. Assume that for all $i = 1, \dots, n$, $\alpha_i = 0$ implies $\alpha_{i+1} = 0$. Suppose that function f in (1) fulfils the following condition:

$$(2) \quad \exists C > 0 \quad |f| \leq C(1 + \sum_{\beta \in A_k} |u_\beta|).$$

If φ is a $[k, \{r_v\}]$ pseudo-oscillating solution of (1) then

$$\sum_{\beta \in A_k} |D^\beta \varphi(x)| \rightarrow 0 \text{ when } |x| \rightarrow 0 \text{ and } x \in [0, \infty)^n.$$

Now we shall present the second theorem. For $i = 1, \dots, n$ let $\{w_v^i\}_{v=1}^\infty$ be sequences of positive real numbers such that $w_v^i \rightarrow \infty$ when $v \rightarrow \infty$, $0 < w_{v+1}^i - w_v^i \rightarrow 0$ when $v \rightarrow \infty$. $W = \{x \in [0, \infty)^n: \exists i = 1, \dots, n, \exists v = 1, 2, \dots, x_i = w_v^i\}$.

We say that the solution φ of (1) is W -oscillating, if

(d) $D^\gamma \varphi$ is defined and continuous in $[0, \infty)^n$, for $0 \leq \gamma \leq \alpha$,

(e) $\varphi \equiv 0$ on W .

Let $p_i: [0, \infty)^n \ni x \rightarrow (w_{v+\alpha_i+1}^i - w_v^i) \in (0, \infty)$

when $w_v^i \leq x_i < w_{v+1}^i$

THEOREM 2. Let φ be a W -oscillating solution of (1) $\alpha \geq (1, \dots, 1)$. We assume that $\exists C: [0, \infty)^n \rightarrow (0, \infty)$ such that

$$(3) \quad |f| \leq C(x)(1 + \sum_{\beta < \alpha} |u_\beta|),$$

$$(4) \quad \forall i = 1, \dots, n \quad C(x)p_i(x) \rightarrow 0$$

when $|x| \rightarrow \infty$ uniformly with respect to $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$.

