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Thin sets in $\mathbf{C}^n$

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1. Introduction

In classical potential theory, a subset E of $\mathbf{R}^n$ is said to be *thin* at a point $x_0 \in \bar{E}$ if there is a subharmonic function φ such that

$$\varlimsup_{\substack{x \to x_0 \\ x \in E \\ x \neq x_0}} \varphi(x) < \varphi(x_0)$$

and E is said to be thin if it is thin at all of its points.

It is well known that E is thin if and only if E is negligible, i.e. there is a uniformly bounded family $(\varphi_i)_{i=1}^{\infty}$ of subharmonic functions such that

$$E \subset \{x \in \mathbf{R}^n;\ \sup_{i \in N} \varphi_i(x) < (\sup_{i \in N} \varphi_i)^*(x)\}$$

where $(\sup_{i \in N} \varphi_i)^*$ is the smallest upper semicontinuous majorant of $\sup_{i \in N} \varphi_i$. Cf. Brelot [2].

By analogy with this, we say that a subset E of $\mathbf{C}^n$ is *thin at* $z_0 \in \bar{E}$ if there is a plurisubharmonic function ψ such that

$$\varlimsup_{\substack{z' \to z_0 \\ z' \in E \\ z' \neq z_0}} \psi(z') < \psi(z_0).$$

We say that E is *thin* if it is thin at all of its points. (Isolated points are considered as thin.)

A set is negligible (cf. Lelong [7]) if there is a family of plurisubharmonic functions $(\varphi_i)_{i \in I}$ locally bounded above such that

$$E \subset \{z \in \mathbf{C}^n;\ \sup_{i \in I} \varphi_i(z) < (\sup_{i \in I} \varphi_i)^*(z)\}$$

In Section 2 we prove that every thin set is negligible. The converse is not true; the set

$$E = \{(z_1, z_2) \in \mathbf{C}^2;\ z_2 = 0\}$$

is negligible but it is not thin at a single point.

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In Section 3 and 4 we discuss connections between L-regularity and non-thinness.

We recall some notation (see Siciak [13] and Cegrell [4, 5]). Denote by PSH(U) the plurisubharmonic functions on U.

If E is a subset of the unit ball B in C^n we put

$$h_E(z) = \sup\{\varphi(z);\ \varphi \in \mathrm{PSH}(B),\ \varphi|_E = -1,\ \varphi \leqslant 0\}$$

and

$$h_E^*(z) = \overline{\lim_{z' \to z}}\ h_E(z').$$

The set E is said to be L-regular at $z_0 \in \bar{E}$ if h_E is continuous at z_0; E is said to be L-regular if it is L-regular at all of its points.

2. Thin sets are negligible

THEOREM 2.1. *Every thin set is negligible.*

Proof. Let E be a thin set in C^n. By Lelong [7], a denumerable union of negligible sets is negligible. We may then assume that E is relatively compact in B where B is a large ball. Cover E with small balls $(B_i)_{i=1}^{\infty}$ so that to any ball P which meets E at z_0 there is an $i \in N$ so that $z_0 \in B_i \subset P$. Put

$$h_i = \sup\{\varphi(z);\ \varphi \in \mathrm{PSH}(B),\ \varphi|_{B_i \cap E} \leqslant -1,\ \varphi \leqslant 0\}$$

and

$$N_i = \{z \in B;\ h_i(z) < h_i^*(z)\}.$$

Then $N = \bigcup_{i=1}^{\infty} N_i$ is negligible and we claim that $E \subset N$.

For given $z_0 \in E$. Then there is a $\psi \in \mathrm{PSH}(B)$ such that $\overline{\lim\limits_{\substack{z' \to z_0 \\ z' \in E \\ z \neq z_0}}}\ \psi(z') < \psi(z_0)$. Then there is a ball P containing z_0 and $\varepsilon > 0$ so that $\psi(z) \leqslant \psi(z_0) - \varepsilon$ on $P \cap E - \{z_0\}$. We may of course assume that $\psi \leqslant 0$ on B. Choose B_i so that $z_0 \in B_i \subset P$ and put

$$\psi_j(z) = \frac{\psi(z)}{-(\psi(z_0) - \varepsilon)} + \frac{1}{j} \log|z - z_0|.$$

We see that $\psi_j|_{B_i \cap E} \leqslant -1$ for every j so

$$h_i^*(z_0) \geqslant \frac{\psi(z_0)}{-(\psi(z_0) - \varepsilon)} > -1 = h_i(z_0)$$

which means that $z_0 \in N_i$. Since z_0 was any point in E we have proved that $E \subset N$.

Remark: By a result of Bedford and Taylor [3], every negligible set in C^n is polar so in particular, every thin set in C^n is polar.

3. Restriction of plurisubharmonic functions to curves

It is well known, that for every subharmonic function φ defined near $z_0 \in \boldsymbol{C}$, we have

$$\varlimsup_{\substack{t \to 0 \\ t > 0}} \varphi(z(t)) = \varphi(z_0) \tag{$*$}$$

for every Jordan curve $[0, 1] \in t \mapsto z(t)$ with $z(0) = z_0$. In other words, a Jordan curve is not thin at any of its points. But in fact, more is true. It follows from Leja's polynomial lemma [6] that if $[0, 1] \in t \mapsto z(t)$ is a non-constant continuous curve then it is not thin at any of its points (cf. Nguyen Thanh Van [8]).

However, in higher dimension, this is not true without further restrictions on the curve.

Ex. Let $(\alpha_i)_{i=1}^{\infty}$ and $(x_i)_{i=1}^{\infty}$ be decreasing sequences of positive numbers converging to zero but such that

$$\sum_{i=1}^{\infty} \alpha_i \log x_i > -\infty .$$

Put

$$\psi(z_1, z_2) = \sum_{i=1}^{\infty} \alpha_i \log|z_1 - x_i|\,|z_2 - x_i| .$$

Then ψ is plurisubharmonic on $\boldsymbol{C}^2$ and

$$(x_i)_{i=1}^{\infty} \times \boldsymbol{R} \cup \boldsymbol{R} \times (x_j)_{j=1}^{\infty}$$

is thin at $(0, 0)$.

We can approximate the functions

$$F(x) = x_i \qquad x \in [x_{i+1}, x_i]$$

and

$$f(x) = x_{i+1}, \qquad x \in [x_{i+1}, x_i]$$

to get smooth functions G and g with the required property. Furthermore, we can arrange things so that $g < G$ for $x \in [0, 1]$. This gives a negative answer to a question in Pleśniak [9, 3:3]. Another example has been given by Sadullaev [12].

But on the other hand, every continuous curve $[0, 1] \ni t \mapsto z(t)$ which is contained in a 1-dimensional analytic manifold is not thin at any of its points. In particular, we have

PROPOSITION 3.2. *Assume that U is plurisubharmonic near z_0 and that F is a holomorphic map $\boldsymbol{C} \supset \omega \xrightarrow{F} \boldsymbol{C}^n$ where ω contains $[0, 1]$. If $F(0) = z_0$ then*

$$U(z_0) = \varlimsup_{\substack{t \to 0 \\ t > 0}} U(F(t)) .$$

PROPOSITION 3.3. *Let E be a subset of $\boldsymbol{C}^n$ and $\boldsymbol{C} \supset \omega \xrightarrow{F} \boldsymbol{C}^n$ a holomorphic map such that $[0, 1] \subset \omega$. If E is L-regular at every point in $\{F(t);\ t \in [0, 1]\}$ then E is L-regular at $F(0)$.*

Proof. We have $V_E^*(F(t)) = -1, t \in [0, 1]$ by assumption. By Proposition 3.2 we have $V_E^*(F(0)) = -1$ so E is L-regular at $F(0)$.

Remark. The above result contains the attainment criterion discussed in Pleśniak [9, 10].

4. L-regular, polynomially convex sets

Let K be a compact subset of the unit ball B. We denote by K_{loc} the points z_0 in K such that $K \cap B(z_0, r)$ is L-regular at z_0 for every $r>0$.

If K is L-regular it follows from Cegrell [5] that $h_{\overline{K_{\text{loc}}}} = h_K$.

Denote by $F(K)$ the functions on K which are restrictions of negative plurisubharmonic functions on B and consider the Choquet boundary Ch K with respect to $F(K)$ (cf. Bauer [1]).

LEMMA 4.4. *Assume that K is polynomially convex. If K is thin at z_0 then $z_0 \in \overline{\mathrm{Ch} K} \setminus K_{\text{loc}}$.*

Proof. It follows from the proof of Theorem 2.1 that K is not thin at any point in K_{loc}. If $z_0 \in K \setminus \overline{\mathrm{Ch} K}$ then, by approximation, we can find a polynomial f such that

$$|f(z_0)| > \sup_{z \in K \cap S(z_0, r)} |f(z)|$$

for $r>0$ small enough. This contradicts Theorem 5.3. in Rossi [11].

THEOREM 4.5. *If K is a polynomially convex and L-regular set then K is not thin at any of its points.*

Proof. If K is thin at z_0, then $z_0 \in \overline{\mathrm{Ch} K} \setminus K_{\text{loc}}$ by Lemma 4.4. But since K is L-regular we have $\mathrm{Ch} K \subset K_{\text{loc}}$ (cf. Cegrell [5, Thm 2.] so we must have $z_0 \in \overline{\mathrm{Ch} K} \setminus \mathrm{Ch} K$.

On the other hand, since K is thin at z_0 there is a plurisubharmonic function θ and a ball $B(z_0, r)$ so that

$$\theta(z) = \begin{cases} -1; & z \in B(z_0, r) \cap K \setminus \{z_0\} \\ 0; & z = z_0 . \end{cases}$$

Choose $0 < r_1 < r$ so that $\theta(z) \leqslant -\frac{3}{4}$ on $\{z;\ d(z, S(z_0, r) \cap K) = r_1\}$. Put $P = \sup_{\xi \in B} \eta(\xi)$ (>0) where

$$\eta(z) = \begin{cases} -\frac{1}{2}; & z \notin B(z_0, r) \\ \sup(-\frac{1}{2}, \theta); & z \in B(z_0, r) . \end{cases}$$

Define χ by

$$\chi(z) = \frac{h_K + 1}{\inf\limits_{d(z, K) = r_1} (h_K + 1)} (1 + P) - \frac{1}{2} .$$

Observe that $\inf\limits_{d(z,K)=r_1} (h_K+1)>0$ since K is polynomially convex. Thus χ is plurisubharmonic and so is ψ where we define ψ as $\sup(\chi, \eta)$ on $\{z \in B(z_0, r);\ d(z, K) \leq r_1\}$ and as χ otherwise. We then have $\psi(z_0) = 0$ and $\psi(z) = -\frac{1}{2}$, $z \in K \setminus \{z_0\}$ which proves that $z_0 \in \mathrm{Ch}\, K$ which is a contradiction and the theorem is proved.

COROLLARY 4.6. *If K is a polynomially convex and L-regular set then $K \setminus K_{loc}$ contains no isolated points (relatively K).*

Proof. If $z_0 \in K \setminus K_{loc}$ then there is an $r>0$ so that $h^*_{K \cap B(z_0,r)}(z_0) > -1$. If z_0 is isolated in $K \setminus K_{loc}$ relatively K then it follows that K is thin at z_0 which contradicts Theorem 4.5.

Remark 1. The result of Theorem 4.5. can be considered as a generalization of (*) in Section 3: If K is a polynomially convex and L-regular set then $\overline{\lim\limits_{\substack{z \to z_0 \\ z \in K \setminus \{z_0\}}}} \varphi(z) = \varphi(z_0)$ for every $z_0 \in K$ and every plurisubharmonic function defined in a neighborhood of K.

Remark 2. The example in Section 3 shows that the assumption K being L-regular cannot be omitted; any compact subset of $R^n \subset C^n$ is polynomially convex.

We finish by generalizing corollary 4.6.

PROPOSITION 4.7. *Let K be a compact and L-regular set. Then $\hat{K} \setminus K_{loc}$ is not thin at any of its points.*

$\hat{K}$ denotes the polynomially convex hull of K)

Proof. It is easily seen, that if two sets are thin at a given point, then their union is also thin at the point. From this and Theorem 4.5. it follows that $\hat{K} \setminus K_{loc}$ can be thin only at points in $\overline{K}_{loc}$ where K_{loc} is not thin. But K_{loc} is thin at every point in $\overline{K}_{loc} \setminus K_{loc}$ which proves the proposition.

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