

A Remark on Modified Euler's Method for Differential Equations in Banach Spaces

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1. Introduction. Let f be a vector field defined on some nonempty open subset of a real (infinite dimensional) Banach space. It is well known that the Cauchy problem

$$(1) \quad \dot{x} = f(t, x) \quad x(0) = y$$

has a unique solution if f is locally lipschitzian. If f is merely continuous even local existence of solutions can fail. However Lasota and Yorke [1], developing some ideas of Orlicz [2], have shown that for "most" f 's in the Banach space of continuous and bounded vector fields the Cauchy problem (1) has solutions. In addition, as it has been established in [3], for "most" f 's in the same space the corresponding sequence of Euler-Cauchy polygons converges to a solution of (1).

Euler's method is the simplest of all methods for approximating solutions of initial value problems. Another method more useful in numerical problems is the so called "improved polygon method" or the "modified Euler method" which allows to approximate solutions with a better accuracy. In this note we extend a result of [3]. In fact we show that the convergence of the modified Euler method is a generic property in the class of all continuous and bounded vector fields. In other words we prove that for almost all (in the sense of the Baire category) continuous and bounded vector fields f the corresponding sequence of polygons converges to a solution of (1).

2. Notations. Let E be a real (infinite dimensional) Banach space with norm $|\cdot|$. If U is a metric space we denote by $B(u, r)$ ($\bar{B}(u, r)$) the open (closed) ball in U with center $u \in U$ and radius $r \geq 0$. In the sequel, when $u = 0$ and $r > 0$, we write \bar{B}_r in place of $\bar{B}(0, r)$. Let $R^+ = [0, +\infty)$. Let $I = [0, a]$ be a compact interval of R . For any map $h: U \rightarrow E$ set $\|h\|_U = \{\sup|h(u)|, u \in U\}$. In the sequel, we write $\|h\|$ instead of $\|h\|_U$, when this does not cause confusion. Let $C(I, \bar{B}_r)$ be the complete metric space of all continuous functions from I into \bar{B}_r with distance $\|h-k\|$, $h, k \in C(I, \bar{B}_r)$

Let $\mathfrak{M} = \{f|f: I \times \bar{B}_r \rightarrow E \text{ is continuous, } \|f\| \leq r\}$ and $\mathcal{L} = \{f \in \mathfrak{M}: f \text{ is locally lipschitzian}\}$. \mathfrak{M} endowed with the metric of the uniform convergence is a metric space. By the result of [1], \mathcal{L} is dense in \mathfrak{M} .

Any function $x^f \in C(I, E)$ satisfying $x^f(t) = y + \int_0^t f(s, x^f(s)) ds$, $t \in I$, is called a solution of (1).

3. Approaching solutions of problem (1)

For each $n \in N$ we set $h = a/n$. We consider a finite subdivision of I such that $0 = t_1^n < t_2^n < \dots < t_{n+1}^n = a$.

The modified Euler's method, applied to problem (1) yields approximations x_1, \dots, x_{n+1} to the values $x(t_1^n), \dots, x(t_{n+1}^n)$ of the solutions at points t_1^n, \dots, t_{n+1}^n .

The values x_1, \dots, x_{n+1} are calculated recursively by the formulas:

$$\begin{aligned} x_1 &= y = x(0) \\ x_{k+1} &= x_k + hf(t_k^n + h/2, x_k + h/2f(t_k^n, x_k)) \quad k = 1, \dots, n \end{aligned}$$

Here h is called the step of the method.

We consider now the problem (1) with $f \in \mathfrak{M}$. For each $n \in N$, we consider the polygonals defined by:

$$u_n^f(t) = y + \int_0^t f\left(\sum_1^n \chi_i^n(s) t_i^n + h/2, \sum_1^n \chi_i^n(s) u_n^f(t_i^n) + h/2f\left(\sum_1^n \chi_i^n(s) t_i^n, \sum_1^n \chi_i^n(s) u_n^f(t_i^n)\right)\right) ds, \quad t \in I.$$

Here for $i = 1, 2, \dots, n$, $\chi_i^n: I \rightarrow I$ denotes the characteristic function of $[t_i^n, t_{i+1}^n)$, while $\chi_n^n: I \rightarrow I$ stands for the characteristic function of $[t_n^n, 1]$.

Under the stated hypotheses each u_n^f is well defined and $u_n^f \in C(I, \bar{B}_r)$. We note explicitly that u_n^f represents the polygonal of the modified Euler's method which corresponds to problem (1) and to subdivision of I of step $h = a/n$.

It is well known that the sequence $\{u_n^f\}$ converges to a solution of (1) for $f \in \mathcal{L}$. We refer to Collatz [4] for estimates concerning the inherent error and the rounding error. We start with some lemmas.

LEMMA 1. [5] — For each $g \in \mathcal{L}$ and any continuous function $w: [0, a] \rightarrow E$, $a > 0$, there exist numbers $L > 0$ and $\eta > 0$ (which depend on g and w) such that:

$$|g(t_1, u_1) - g(t_2, u_2)| \leq L[|t_1 - t_2| + |u_1 - u_2|]$$

for all (t_i, u_i) such that $\max\{|t_i - t|, |u_i - w(t)|\} < \eta$ $i = 1, 2$.

LEMMA 2. Let $g \in \mathcal{L}$. Let x^g be the unique solution of the problem

$$\dot{x} = g(t, x) \quad x(0) = y$$

defined on I . Then for every $\varepsilon > 0$ there exist $\delta_g(\varepsilon) > 0$ and $\sigma_g(\varepsilon) > 0$ such that if $f \in \bar{B}(g, \delta_g(\varepsilon))$ and $h < \sigma_g(\varepsilon)$ we have

$$(2) \quad \|u_n^f - x^g\| \leq \varepsilon.$$