

A Remark on Modified Euler's Method for Differential Equations in Banach Spaces

Sandra GIUNTINI

1. Introduction. Let f be a vector field defined on some nonempty open subset of a real (infinite dimensional) Banach space. It is well known that the Cauchy problem

$$(1) \quad \dot{x} = f(t, x) \quad x(0) = y$$

has a unique solution if f is locally lipschitzian. If f is merely continuous even local existence of solutions can fail. However Lasota and Yorke [1], developing some ideas of Orlicz [2], have shown that for "most" f 's in the Banach space of continuous and bounded vector fields the Cauchy problem (1) has solutions. In addition, as it has been established in [3], for "most" f 's in the same space the corresponding sequence of Euler-Cauchy polygons converges to a solution of (1).

Euler's method is the simplest of all methods for approximating solutions of initial value problems. Another method more useful in numerical problems is the so called "improved polygon method" or the "modified Euler method" which allows to approximate solutions with a better accuracy. In this note we extend a result of [3]. In fact we show that the convergence of the modified Euler method is a generic property in the class of all continuous and bounded vector fields. In other words we prove that for almost all (in the sense of the Baire category) continuous and bounded vector fields f the corresponding sequence of polygons converges to a solution of (1).

2. Notations. Let E be a real (infinite dimensional) Banach space with norm $|\cdot|$. If U is a metric space we denote by $B(u, r)$ ($\bar{B}(u, r)$) the open (closed) ball in U with center $u \in U$ and radius $r \geq 0$. In the sequel, when $u = 0$ and $r > 0$, we write \bar{B}_r in place of $\bar{B}(0, r)$. Let $R^+ = [0, +\infty)$. Let $I = [0, a]$ be a compact interval of R . For any map $h: U \rightarrow E$ set $\|h\|_U = \{\sup|h(u)|, u \in U\}$. In the sequel, we write $\|h\|$ instead of $\|h\|_U$, when this does not cause confusion. Let $C(I, \bar{B}_r)$ be the complete metric space of all continuous functions from I into \bar{B}_r with distance $\|h-k\|$, $h, k \in C(I, \bar{B}_r)$

Let $\mathfrak{M} = \{f|f: I \times \bar{B}_r \rightarrow E \text{ is continuous, } \|f\| \leq r\}$ and $\mathcal{L} = \{f \in \mathfrak{M}: f \text{ is locally lipschitzian}\}$. \mathfrak{M} endowed with the metric of the uniform convergence is a metric space. By the result of [1], \mathcal{L} is dense in \mathfrak{M} .

Any function $x^f \in C(I, E)$ satisfying $x^f(t) = y + \int_0^t f(s, x^f(s)) ds$, $t \in I$, is called a solution of (1).

3. Approaching solutions of problem (1)

For each $n \in N$ we set $h = a/n$. We consider a finite subdivision of I such that $0 = t_1^n < t_2^n < \dots < t_{n+1}^n = a$.

The modified Euler's method, applied to problem (1) yields approximations x_1, \dots, x_{n+1} to the values $x(t_1^n), \dots, x(t_{n+1}^n)$ of the solutions at points t_1^n, \dots, t_{n+1}^n .

The values x_1, \dots, x_{n+1} are calculated recursively by the formulas:

$$\begin{aligned} x_1 &= y = x(0) \\ x_{k+1} &= x_k + hf(t_k^n + h/2, x_k + h/2f(t_k^n, x_k)) \quad k = 1, \dots, n \end{aligned}$$

Here h is called the step of the method.

We consider now the problem (1) with $f \in \mathfrak{M}$. For each $n \in N$, we consider the polygonals defined by:

$$u_n^f(t) = y + \int_0^t f\left(\sum_1^n \chi_i^n(s) t_i^n + h/2, \sum_1^n \chi_i^n(s) u_n^f(t_i^n) + h/2f\left(\sum_1^n \chi_i^n(s) t_i^n, \sum_1^n \chi_i^n(s) u_n^f(t_i^n)\right)\right) ds, \quad t \in I.$$

Here for $i = 1, 2, \dots, n$, $\chi_i^n: I \rightarrow I$ denotes the characteristic function of $[t_i^n, t_{i+1}^n)$, while $\chi_n^n: I \rightarrow I$ stands for the characteristic function of $[t_n^n, 1]$.

Under the stated hypotheses each u_n^f is well defined and $u_n^f \in C(I, \bar{B}_r)$. We note explicitly that u_n^f represents the polygonal of the modified Euler's method which corresponds to problem (1) and to subdivision of I of step $h = a/n$.

It is well known that the sequence $\{u_n^f\}$ converges to a solution of (1) for $f \in \mathcal{L}$. We refer to Collatz [4] for estimates concerning the inherent error and the rounding error. We start with some lemmas.

LEMMA 1. [5] — For each $g \in \mathcal{L}$ and any continuous function $w: [0, a] \rightarrow E$, $a > 0$, there exist numbers $L > 0$ and $\eta > 0$ (which depend on g and w) such that:

$$|g(t_1, u_1) - g(t_2, u_2)| \leq L[|t_1 - t_2| + |u_1 - u_2|]$$

for all (t_i, u_i) such that $\max\{|t_i - t|, |u_i - w(t)|\} < \eta$ $i = 1, 2$.

LEMMA 2. Let $g \in \mathcal{L}$. Let x^g be the unique solution of the problem

$$\dot{x} = g(t, x) \quad x(0) = y$$

defined on I . Then for every $\varepsilon > 0$ there exist $\delta_g(\varepsilon) > 0$ and $\sigma_g(\varepsilon) > 0$ such that if $f \in \bar{B}(g, \delta_g(\varepsilon))$ and $h < \sigma_g(\varepsilon)$ we have

$$(2) \quad \|u_n^f - x^g\| \leq \varepsilon.$$

Proof. Let $L > 1$ and $\eta > 0$ correspond to x^g according to lemma 1. Let $\varepsilon > 0$. We claim that the statement is true if we take $\delta_g(\varepsilon) = \delta$ and $\sigma_g(\varepsilon) = \sigma$, where

$$\delta = \min(\eta e^{-La}/8a, \varepsilon e^{-La}/a)$$

$$\sigma = \min(\eta e^{-La}/4Lra, 2\varepsilon e^{-La}/Lra).$$

We fix $n \in N$ and take $f \in B(g, \delta)$. Define

$$J = \{t \in I: \|u_n^f - x^g\|_{[0,t]} \leq \eta/4\}.$$

The set J is nonempty and closed. We put $b = \sup J$. We claim that $b = a$. To see this we suppose the contrary, that is $b < a$, and we take $0 < \tau < a - b$.

In order to prove (2) we consider the following difference

$$u_n^f(t) - x^g(t) = y + \int_0^t f\left(\sum_1^n \chi_i^n(s) t_i^n + h/2, \sum_1^n \chi_i^n(s) u_n^f(t_i^n) + h/2 f\left(\sum_1^n \chi_i^n(s) t_i^n, \sum_1^n \chi_i^n(s) u_n^f(t_i^n)\right)\right) ds - y - \int_0^t g(s, x^g(s)) ds \quad \text{for } t \in [0, b + \tau]$$

from which adding and subtracting on the right hand side the quantities

$$\int_0^t g\left(\sum_1^n \chi_i^n(s) t_i^n + h/2, \sum_1^n \chi_i^n(s) u_n^f(t_i^n) + h/2 f\left(\sum_1^n \chi_i^n(s) t_i^n, \sum_1^n \chi_i^n(s) u_n^f(t_i^n)\right)\right) ds,$$

we obtain

$$|u_n^f(t) - x^g(t)| \leq \int_0^t |f\left(\sum_1^n \chi_i^n(s) t_i^n + h/2, \sum_1^n \chi_i^n(s) u_n^f(t_i^n) + h/2 f\left(\sum_1^n \chi_i^n(s) t_i^n, \sum_1^n \chi_i^n(s) u_n^f(t_i^n)\right)\right) - g\left(\sum_1^n \chi_i^n(s) t_i^n + h/2, \sum_1^n \chi_i^n(s) u_n^f(t_i^n) + h/2 f\left(\sum_1^n \chi_i^n(s) t_i^n, \sum_1^n \chi_i^n(s) u_n^f(t_i^n)\right)\right)| ds + \int_0^t |g\left(\sum_1^n \chi_i^n(s) t_i^n + h/2, \sum_1^n \chi_i^n(s) u_n^f(t_i^n) + h/2 f\left(\sum_1^n \chi_i^n(s) t_i^n, \sum_1^n \chi_i^n(s) u_n^f(t_i^n)\right)\right) - g(s, x^g(s))| ds \leq \delta a + A(t)$$

where $A(t)$ denotes the last integral. Since

$$\left| \sum_1^n \chi_i^n(s) u_n^f(t_i^n) + h/2 f\left(\sum_1^n \chi_i^n(s) t_i^n, \sum_1^n \chi_i^n(s) u_n^f(t_i^n)\right) - x^g(s) \right| \leq \left| \sum_1^n \chi_i^n(s) u_n^f(t_i^n) - x^g(s) \right| + h/2 \left| f\left(\sum_1^n \chi_i^n(s) t_i^n, \sum_1^n \chi_i^n(s) u_n^f(t_i^n)\right) \right| \leq \left| \sum_1^n \chi_i^n(s) u_n^f(t_i^n) - x^g(s) \right| + \sigma r/2$$

and

$$\begin{aligned} \left| \sum_1^n \chi_i^n(s) u_n^f(t_i^n) - x^g(s) \right| &\leq \left| \sum_1^n \chi_i^n(s) u_n^f(t_i^n) - \sum_1^n \chi_i^n(b) u_n^f(t_i^n) \right| + \\ &\left| \sum_1^n \chi_i^n(b) u_n^f(t_i^n) - x^g(b) \right| + |x^g(b) - x^g(s)| \leq \eta \end{aligned}$$

then by definition of characteristic function

$$|u_n^f(t) - x^g(t)| \leq \delta a + Lra\sigma/2 + L \int_0^t |u_n^f(s) - x^g(s)| ds.$$

From this, by the Gronwall's inequality

$$(3) \quad |u_n^f(t) - x^g(t)| \leq (\delta a + Lra\sigma/2) e^{Lt}, \quad t \in [0, b + \tau).$$

Since $\delta \leq \eta e^{-La}/8a$ and $\sigma \leq \eta e^{-La}/4Lra$ the inequality (3) implies $b + \tau \in J$, and this is a contradiction. Thus $b = a$ and (3) is satisfied for each $t \in [0, a]$. By the choice of δ and σ we obtain from (3) the statement. The following lemma is implicitly contained in [1].

LEMMA 3. Let \mathcal{H} be a complete metric space. Let Γ be a dense subset of \mathcal{H} . Let there exist a function $\varphi: \mathcal{H} \rightarrow R^+$ satisfying the condition for each $x \in \Gamma$ and any sequence $\{x_n\} \subset \mathcal{H}$ which converges to x we have $\lim_{n \rightarrow +\infty} \varphi(x_n) = 0$. Then the set $\{x \in \mathcal{H} : \varphi(x) = 0\}$ is residual in \mathcal{H} .

(A residual set in \mathcal{H} is the complement of a set of Baire first category).

THEOREM 1. Let \mathfrak{M}^0 be the subset of \mathfrak{M} of all those $f \in \mathfrak{M}$ such that the corresponding sequence $\{u_n^f\} \subset C(I, \bar{B}_r)$ of the polygonals of the modified Euler's method converges uniformly on I to a solution x^f of (1). Then \mathfrak{M}^0 is a residual set in \mathfrak{M} .

Proof. Following [3] we define $\varphi: \mathfrak{M} \rightarrow R^+$ by:

$$\varphi(f) = \limsup_{p, q \rightarrow +\infty} \|u_p^f - u_q^f\|.$$

We shall see that the assumptions of the previous lemma are satisfied with $\mathcal{H} = \mathfrak{M}$, $\Gamma = \mathcal{L}$.

Indeed, let $\varepsilon > 0$. Let $g \in \mathcal{L}$ and x^g be the unique solution of problem (1) defined on I . Let $\delta_g(\varepsilon/2)$ and $\sigma_g(\varepsilon/2)$ correspond according to lemma 2. Let $h = a/n$ be arbitrary, then for any $f \in \bar{B}(g, \delta_g(\varepsilon/2))$ and p, q sufficiently large, say $p, q \geq \bar{n}$, we have

$$\|u_p^f - u_q^f\| \leq \|u_p^f - x^g\| + \|u_q^f - x^g\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus

$$\limsup_{p, q \rightarrow +\infty} \|u_p^f - u_q^f\| \leq \varepsilon$$

and since the subdivision of I is arbitrary we have

$$\varphi(f) \leq \varepsilon \quad \text{for } f \in \bar{B}(g, \delta_g(\varepsilon/2)).$$

Hence the set $\mathfrak{M}^* = \{f \in \mathfrak{M} : \varphi(f) = 0\}$ is residual in \mathfrak{M} . Since $\mathfrak{M}^* \subset \mathfrak{M}^0$ the proof is complete.

References

- [1] A. Lasota, J. Yorke, *The generic property of existence of solutions of differential equations in Banach space*, J. Differential Equations, 13 (1973), 1-12.
- [2] M. W. Orlicz, *Zur Theorie der Differentialgleichung $y' = f(x, y)$* , Acad. des Sciences, Bull. Inter. Sci. Math. (1932A), 221-228.
- [3] F. S. De Blasi, J. Myjak, *Generic flows generated by continuous vector fields in Banach spaces*, C. R. Acad. Sci. Paris Sér. A 287 (1978).
- [4] L. Collatz, *The numerical treatment of differential equations*, Die Grundle. der Math. Wissens., Springer Verlag, Berlin (1960).
- [5] B. S. De Blasi, J. Myjak, *Generic properties of differential equations in a Banach space*, Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys. (14) 26 (1978), 287-292.

Received January 30, 1981