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# Holomorphic Continuation with Restricted Growth

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**1. Introduction.** In the paper we shall present a generalization of the following theorem to the case of Riemann domains.

THEOREM N ([3]). *Let $M$ be an $n-1$ dimensional analytic submanifold of $\boldsymbol{C}^n$ such that there exists a function $G \in \mathcal{O}(\boldsymbol{C}^n)$ for which $M = G^{-1}(0)$ and*

$$|G(z)| \leqslant e^{a(\|z\|^\sigma+1)}, \quad z \in \boldsymbol{C}^n,$$

$$\max\left\{\left|\frac{\partial G}{\partial z_k}(z)\right| : k = 1, \ldots, n\right\} \geqslant e^{-b(\|z\|^\sigma+1)}, \quad z \in M,$$

*where $\sigma > 0$, $a, b \geqslant 1$ are constants.*

*Then for every $m \geqslant 1$ there exists $\hat{m} \geqslant 1$ (depending only on $n, \sigma, a, b, m$) such that every function $f \in \mathcal{O}(M)$ satisfying the condition*

$$|f(z)| \leqslant e^{m(\|z\|^\sigma+1)}, \quad z \in M,$$

*admits an extension $\hat{f} \in \mathcal{O}(\boldsymbol{C}^n)$ such that*

$$|\hat{f}(z)| \leqslant e^{\hat{m}(\|z\|^\sigma+1)}, \quad z \in \boldsymbol{C}^n.$$

Let $(X, p)$ be a Riemann domain spread over $\boldsymbol{C}^n$, $p = (p_1, \ldots, p_n)$: $X \to \boldsymbol{C}^n$. We say that $(X, p)$ is a Stein domain if $X$ with the natural analytic structure given by $p$ is a Stein manifold.

From now on $(X, p)$ *will always denote a Riemann-Stein domain over* $\boldsymbol{C}^n$, in particular, $(X, p) = (\Omega, \mathrm{id}_\Omega)$, where $\Omega$ is a domain of holomorphy in $\boldsymbol{C}^n$.

For $x \in X$, $r > 0$, let $\hat{B}(x, r)$ denote an open neighbourhood of $x$ mapped homeomorphically by $p$ onto the Euclidean ball $B(p(x), r) \subset \boldsymbol{C}^n$. Let us put:

$$\varrho(x) = \sup\{r > 0 : \hat{B}(x, r) \text{ exists}\},$$

$$\hat{B}(x) = \hat{B}(x, \varrho(x)),$$

$$\delta_X = \min\{\varrho, (1 + \|p\|^2)^{-1/2}\}.$$

A function $\delta: X \to (0, 1]$ is said to be *a weight function on $X$* if

(1) $\qquad \delta \leqslant \delta_X,$

(2) $\qquad |\delta(x) - \delta(x')| \leqslant \|p(x) - p(x')\|, \quad x \in X, x' \in \hat{B}(x)$ (comp. [1]).

Observe that (2) implies

(3) $$|\delta(x')-\delta(x'')|\leqslant\|p(x')-p(x'')\|,\ x',x''\in\hat{B}(x,\tfrac{1}{3}\varrho(x)),\ x\in X,$$

(4) $$(1-\theta)\delta(x)\leqslant\delta(x')\leqslant(1+\theta)\delta(x),\ x\in X,\ x'\in\hat{B}(x,\theta\delta(x)),\ 0<\theta<1.$$

Let us remark that if $(X,p)=(\Omega,\mathrm{id}_\Omega)$ and if $\delta$ is a weight function on $\Omega$ then in view of (1)

(5) $$\int_X \delta^{2(n+\varepsilon)}d\lambda\leqslant\int_{C^n}(1+\|z\|^2)^{-(n+\varepsilon)}d\lambda\leqslant\frac{c(n)}{\varepsilon},\quad \varepsilon>0,$$

where $\lambda$ denotes the Lebesgue measure in $C^n$.

Let $\mu$ denote the measure on $X$ generated by the volume element $(2i)^{-n}d\bar{p}_1\wedge\ldots\wedge d\bar{p}_n\wedge dp_1\wedge\ldots\wedge dp_n$. Put

$H^{(s)}(X,\delta)=\{f\in\mathcal{O}(X)\colon \|\delta^s f\|_2=(\int_X|f|^2\delta^{2s}d\mu)^{1/2}<+\infty\}$,

$\mathcal{O}^{(s)}(X,\delta)=\{f\in\mathcal{O}(X)\colon \|\delta^s f\|_\infty<+\infty\}$, and analogously for a submanifold $M$ of $X$: $\mathcal{O}^{(s)}(M,\delta)=\{f\in\mathcal{O}(M)\colon \|\delta^s f\|_\infty<+\infty\}$, $s\geqslant 0$.

The following three results will be useful in the sequel.

*Let $\delta$ be a weight function on $X$. Then:*

(6) ([1], Prop. 2) $\quad \|\delta^{s+|\alpha|}\partial^\alpha f\|_\infty\leqslant\alpha!\sqrt{n}^{|\alpha|}2^{s+|\alpha|}\|\delta^s f\|_\infty,\ f\in\mathcal{O}(X),\ s\geqslant 0,\ \alpha\in Z^n_+$.

(7) ([1], Prop. 3) $\quad \|\delta^{s+n}f\|_\infty\leqslant[(1-\theta)^s\theta^n\sqrt{\tau_n}]^{-1}\|\delta^s f\|_2,\ f\in\mathcal{O}(X),\ s\geqslant 0,\ 0<\theta<1,$

*where $\tau_n$ denotes the volume of the unit ball in $C^n$.*

(8) ([1], Th. 2) *If moreover $-\log\delta\in\mathrm{PSH}(X)$, then for every $\bar{\partial}$-closed form $u\in L^2_{(0,1)}(X,\mathrm{loc})$ there exists $v\in L^2(X,\mathrm{loc})$ such that $\bar{\partial}v=u$ and $\|\delta^{s+2}v\|_2\leqslant\|\delta^s u\|_2=(\int_X|u|^2\delta^{2s}d\mu)^{1/2}$.*

The main result of the paper is the following

THEOREM 1. *Let $(X,p)$ be a Riemann-Stein domain over $C^n$ and let $\delta$ be a weight function on $X$ such that $-\log\delta\in\mathrm{PSH}(X)$ and, for some $\alpha_0\geqslant 0$, $A_0=\|\delta^{\alpha_0}\|_2<+\infty$*[1]*. Let $M$ be an $n-1$ dimensional analytic submanifold of $X$ such that there exists a function $G\in\mathcal{O}(X,\delta)$ for which $M=G^{-1}(0)$ and*

$$A=\|\delta^\alpha G\|_\infty<+\infty,$$

$$\max\left\{\left|\frac{\partial G}{\partial x_k}(x)\right|\colon k=1,\ldots,n\right\}\geqslant B\delta^\beta(x),\ x\in M,$$

*where $\alpha,\beta\geqslant 0$, $B>0$ are constants.*

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[1] Comp. (5) and note that if $\Omega$ is bounded then we can put $\alpha_0=0$.

*Then for every* $\eta > 1$ *there exists a constant* $c_0 > 0$ *(depending only on* $n, \alpha, \beta, A_0, A, B, \eta$*) such that:*

*for every* $s \geqslant 0$ *there exists a linear continuous extension operator*

$$L_s\colon \mathcal{O}^{(s)}(M, \delta) \to H^{(s+\gamma_0)}(X, \delta)$$

*such that*

$$L_s(f) = f \text{ on } M,\ f \in \mathcal{O}^{(s)}(M, \delta),$$

$$\|L_s\| \leqslant c_0 \eta^s,$$

*where* $\gamma_0 = \alpha_0 + 5\alpha + 5\beta + 8$.

Notice that $\gamma_0$ is effectively given and, as it will follow from the proof, $c_0$ may be also effectively calculated.

The proof will be presented in Section 2.

COROLLARY 1. *Under the assumptions of Theorem* 1, *for every* $\eta > 1$ *there exists a constant* $c > 0$ *(depending only on* $n, \alpha, \beta, A_0, A, B, \eta$*) such that:*

*for every* $s \geqslant 0$ *there exists a linear continuous extension operator* $L_s\colon \mathcal{O}^{(s)}(M, \delta) \to \mathcal{O}^{(s+\gamma)}(X, \delta)$ *such that*

$$L_s(f) = f \text{ on } M,\ f \in \mathcal{O}^{(s)}(M, \delta),$$

$$\|L_s\| \leqslant c\eta^s,$$

*where* $\gamma = \gamma_0 + n$ $(= n + \alpha_0 + 5\alpha + 5\beta + 8)$.

Proof. Let us fix $\eta_0 > 1$ and let $0 < \theta < 1$ be chosen such that $\eta = (1-\theta)\eta_0 > 1$. Let $c_0, (L_s)_{s \geqslant 0}$ be associated with $\eta$ accordingly to Theorem 1. In view of (7), $L_s$ may be regarded as a linear continuous operator of $\mathcal{O}^{(s)}(M, \delta)$ into $\mathcal{O}^{(s+\gamma_0+n)}(X, \delta)$ and as an operator between these spaces, it has the norm $\leqslant c\eta_0^s$, where $c = c_0[(1-\theta)^{\gamma_0}\theta^n\sqrt{\tau_n}]^{-1}$. The proof is finished.

COROLLARY 2. *Theorem N is a consequence of Corollary* 1.

Proof. Let $M, G, \sigma, a, b$ be as in Th. N. It is easy to show that for some $\varkappa = \varkappa(\sigma) > 0$ the function

$$\delta(z) = \varkappa \min\{e^{-1}, e^{-\|z\|^\sigma}\},\ z \in \boldsymbol{C}^n,$$

is a weight function in $\boldsymbol{C}^n$. Clearly $-\log\delta \in \mathrm{PSH}(\boldsymbol{C}^n)$.

Thus $(\boldsymbol{C}^n, \mathrm{id}_{\boldsymbol{C}^n})$, $\delta$, $M$, $G$ satisfy all the assumptions of Th. 1 with $\alpha_0 = n + \varepsilon$ (comp. (5)), $\alpha = a$, $\beta = b$.

Let $c, (L_s)_{s \geqslant 0}$ be associated with $\eta = 2$ accordingly to Corollary 1. Fix $m \leqslant 1$ and $f \in \mathcal{O}(M)$ with $|f(z)| \leqslant e^{m(\|z\|^\sigma + 1)}$, $z \in M$. Then $f \in \mathcal{O}^{(m)}(M, \delta)$ and $\|\delta^m f\|_\infty \leqslant (\varkappa e)^m$. Put $\hat{f} = L_m(f)$. It is seen that $\hat{f} \in \mathcal{O}(\boldsymbol{C}^n)$, $\hat{f} = f$ on $M$ and $\|\delta^{m+\gamma}\hat{f}\|_\infty \leqslant c(2\varkappa e)^m$, where $\gamma = 2n + 5a + 5b + 8 + \varepsilon$. Putting $\hat{m} = m + \gamma + \log^+\left[\frac{(2e)^m}{\varkappa^\gamma}\right]$ we get $|\hat{f}(z)| \leqslant e^{\hat{m}(\|z\|^\sigma + 1)}$, $z \in \boldsymbol{C}^n$, which finishes the proof.

Below we shall present two (in some sense extremal) examples of a domain of holomorphy $\Omega\subset C^n$ and an $n-1$ dimensional analytic submanifold $M\subset\Omega$ such that *for every* weight function $\delta$ in with $-\log\delta\in\mathrm{PSH}(\Omega)$ the assumptions of Theorem 1 are fulfilled.

a) $\Omega$ is a bounded domain of holomorphy in $C^n$, $M=M_0\cap\Omega$, where $M_0=G_0^{-1}(0)$, $G_0\in\mathcal{O}(\Omega_0)$, $\bar{\Omega}\subset\Omega_0\in\operatorname{top}C^n$, and $d_zG_0\neq 0$, $z\in M_0$.

In this case $\alpha_0=\alpha=\beta=0$.

b) $\Omega=C^n$, $M=G^{-1}(0)$, where $G$ is a polynomial of $n$ complex variables such that $d_zG\neq 0$, $z\in M$.

In this case we only need to verify that there exist $k\geqslant 0$, $c>0$ such that

$$(1+\|z\|)^k\|d_zG\|>c,\ z\in M.$$

The method of the proof is due to L. Lempert.

The polynomials $G,\dfrac{\partial G}{\partial z_1},\dots,\dfrac{\partial G}{\partial z_n}$ have no common zeros in $C^n$, hence there exist polynomials $P_0,P_1,\dots,P_n$ such that

$$P_0G+\sum_{j=1}^{n}P_j\frac{\partial G}{\partial z_j}\equiv 1\ \text{ in } C^n.$$

In consequence, for $z\in M$ we get

$$1=\sum_{j=1}^{n}P_j(z)\frac{\partial G}{\partial z_j}(z)\leqslant\|(P_1(z),\dots,P_n(z))\|\,\|d_zG\|\leqslant\mathrm{const}.(1+\|z\|)^k\|d_zG\|,$$

where $k=\max\{\deg P_j\colon j=1,\dots,n\}$.

## 2. Proof of Theorem 1.

The space $H^{(s+\gamma_0)}(X,\delta)$ is a Hilbert space whose topology is stronger than the topology of uniform convergence on compact subsets of $X$, hence in view of Lemma 1 in [2], it is sufficient to prove the following slightly weaker version of Theorem 1.

THEOREM 1′. *Under the assumptions of Theorem* 1, *for every* $\eta>1$ *there exists a constant* $c_*=c_*(n,\alpha,\beta,A_0,A,B,\eta)>0$ *such that: for every* $s\geqslant 0$, $f\in\mathcal{O}^{(s)}(M,\delta)$ *there exists* $\hat{f}\in H^{(s+\gamma_0)}(X,\delta)$ *with* $\hat{f}=f$ *on* $M$ *and* $\|\delta^{s+\gamma_0}\hat{f}\|_2\leqslant c_*\|\delta^s f\|_\infty{}^2$.

Proof of Theorem 1′. Without loss of generality we may assume that $A\geqslant 1$, $B=1$.

For the proof, analogously as in [3], we shall construct some special open coverings $(U_i)_i$ of $M$, holomorphic retractions $\pi_i\colon U_i\to U_i\cap M$ and a partition of unity, namely:

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[1] We can put $c_0=2c_*$.

PROPOSITION 1. *There exists* $N$, $1 \leqslant N \leqslant n$, *such that for every* $\eta > 1$ *there exists an open covering* $U_0, U_1, \ldots, U_N$ *of* $X$, $U_0 \cap M = \emptyset$, $U_k \cap M \neq \emptyset$, $k = 1, \ldots, N$, *holomorphic retractions* $\pi_k\colon U_k \to U_k \cap M$, $k = 1, \ldots, N$, *a partition of unity* $\xi_0, \xi_1, \ldots, \xi_N \in C(X, [0, 1])$ *and constants* $C_j = C_j(n, \alpha, \beta, A, \eta) > 0$, $j = 1, 2$ *such that:*

(i) $\operatorname{supp} \xi_k \subset U_k$, $\bar{\partial}\xi_k \in L^2_{(0,1)}(X, \mathrm{loc})$,

(9)
$$\delta^{2\alpha+2\beta+3}|\bar{\partial}\xi_k| \leqslant C_1, \quad k = 0, \ldots, N;$$

(ii) *given* $f \in \mathcal{O}^{(s)}(M, \delta)$, *if* $f_0 = 0$, $f_k = f \circ \pi_k$, $k = 1, \ldots, N$, *then*

(10)
$$\delta^s|f_k| \leqslant \eta^s \|\delta^s f\|_\infty \ \textit{in}\ U_k, \quad k = 0, \ldots, N,$$
$$\delta^{s+2\alpha+2\beta+3}|f_l - f_k| \leqslant C_2 \eta^s \|\delta^s f\|_\infty |G| \ \textit{in}\ U_k \cap U_l,$$

$k, l = 0, \ldots, N.$

Assuming this result for a moment, we shall finish the main proof of Theorem 1′.

Let us fix $\eta > 1$ and $f \in \mathcal{O}^{(s)}(M, \delta)$. Put $L = \|\delta^s f\|_\infty$. Let $(U_k)_{k=0}^N$, $(\pi_k)_{k=1}^N$, $(\xi_k)_{k=0}^N$, $C_1$, $C_2$ and $(f_k)_{k=0}^N$ be as in Proposition 1.

Define $f_{kl} = \dfrac{f_l - f_k}{G}$ in $U_k \cap U_l$ (note that $f_{kl} \in \mathcal{O}(U_k \cap U_l)$) and let $b_l\colon U_l \to \mathbf{C}$ ($l = 0, \ldots, N$) be given by the formula

$$b_l = \sum_{k=0}^{N} \xi_k f_{kl},$$

where we mean that $\xi_k f_{kl} = 0$ in $U_l \setminus \operatorname{supp} \xi_k$, $k = 0, \ldots, N$.

Clearly $b_l \in C(U_l)$ and, in view of (10),

$$\delta^{s+2\alpha+3\beta+3}|b_l| \leqslant (n+1)\, C_2 \eta^s L.$$

By dint of (9), $\bar{\partial} b_l \in L^2_{(0,1)}(U_l, \mathrm{loc})$ and

$$\delta^{s+4\alpha+5\beta+3}|\bar{\partial} b_l| \leqslant (n+1)\, C_1 C_2 \eta^s L.$$

It is seen that $b_l - b_k = f_{kl}$ in $U_k \cap U_l$. In particular, the form $u$ given by the formula $u = \bar{\partial} b_l$ in $U_l$, $l = 0, \ldots, N$, is a well-defined $\bar{\partial}$-closed form of the class $L^2_{(0,1)}(X, \mathrm{loc})$ with

$$\|\delta^{s+\alpha_0+4\alpha+5\beta+6} u\|_2 \leqslant (n+1) A_0 C_1 C_2 \eta^s L.$$

Hence, by (8), there exists $v \in L^2(X, \mathrm{loc})$ such that $\bar{\partial} v = u$ and

$$\|\delta^{s+\alpha_0+4\alpha+5\beta+8} v\|_2 \leqslant (n+1) A_0 C_1 C_2 \eta^s L.$$

Now let $\hat{f} = f_l - G(b_l - v)$ in $U_l$, $l = 0, \ldots, N$. It is clear that $\hat{f}$ is well-defined holomorphic on $X$ and $\hat{f} = f$ on $M$. It remains to estimate the growth of $\hat{f}$.

$$\|\delta^{s+\gamma_0}\hat{f}\|_2^2 \leqslant 2 \sum_{l=0}^{N} \Big[\int_{U_l} (\delta^s|f_l|)^2 \delta^{2\alpha_0} d\mu + 2\int_{U_l} (\delta^\alpha|G|)^2 (\delta^{s+2\alpha+3\beta+3}|b_l|)^2 \delta^{2\alpha} d\mu +$$
$$+ 2\int_{U_l} (\delta^\alpha|G|)^2 |v|^2 \delta^{2(s+\alpha_0+4\alpha+5\beta+8)} d\mu\Big]$$
$$\leqslant 2(n+1) A_0^2 [1 + 2A^2(n+1)^2(1+C_1^2) C_2^2] \eta^{2s} L^2.$$

The proof of Theorem 1′ is finished.

Proof of Proposition 1. I. Local retractions.

We start with a generalization of Lemma 6 in [3]. Put $t = t(\beta) = \frac{1}{2}(\frac{2}{3})^\beta$ and let

$$M_k^j = \left\{x \in M: \left|\frac{\partial G}{\partial x_k}(x)\right| > t^j \delta^\beta(x)\right\}, \qquad k = 1, \dots, n,\ j = 1, 2, \dots$$

Without loss of generality we may assume that, for some $1 \leqslant N \leqslant n$, $M = \bigcup_{k=1}^{N} M_k^1$ and $M_k^1 \neq \emptyset$, $k = 1, \dots, N$.

For $x \in X$, $0 < c \leqslant 1$ and $\gamma \geqslant 1$ let

$$\Delta_k(x;\ c, \gamma) = \{y \in \hat{B}(x, c\delta^\gamma(x)):\ p_j(y) = p_j(x),\ j = 1, \dots, k-1, k+1, \dots, n\},\ k = 1, \dots, n.$$

Define $\gamma_1 = \alpha + \beta + 2$, $c_1 = t^8(nA2^{\alpha+4})^{-1}$.

LEMMA 1. *For every* $x \in M_k^j$, $y \in \Delta_k(x;\ c_1, \gamma_1)$:

$$|G(y)| \geqslant \frac{t^j}{2} \delta^\beta(x) |p_k(y) - p_k(x)|, \quad k = 1, \dots, n,\ j = 1, \dots, 8.$$

*In particular, for every* $x \in M_k^j$: $\Delta_k(x;\ c_1, \gamma_1) \cap M = \{x\}$, $k = 1, \dots, n$, $j = 1, \dots, 8$.

Proof. Observe that $G(y) = \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m G}{\partial x_k^m}(x)[p_k(y) - p_k(x)]^m$, hence, in view of (6),

$$|G(y)| \geqslant \left|\frac{\partial G}{\partial x_k}(x)[p_k(y) - p_k(x)]\right| - \sum_{m=2}^{\infty} \sqrt{n}^{m} 2^{\alpha+m} A \delta^{-(\alpha+m)}(x) |p_k(y) - p_k(x)|^m$$

$$\geqslant [t^j \delta^\beta(x) - nA2^{\alpha+3}\delta^{-(\alpha+2)}(x)|p_k(y) - p_k(x)|]\,|p_k(y) - p_k(x)|$$

$$\geqslant [t^j \delta^\beta(x) - nA2^{\alpha+3} c_1 \delta^\beta(x)]\,|p_k(y) - p_k(x)| \geqslant \frac{t^j}{2} \delta^\beta(x) |p_k(y) - p_k(x)|.$$

The proof of Lemma 1 is finished.

Let us fix $\eta > 1$ and let $0 < \theta < 1$ be such that $(1+\theta)^2(1-\theta)^{-1} < \eta$.
Put $c_2 = c_1\theta(3^{\gamma_1+1})^{-1}$.

LEMMA 2 (comp. [3], (v)). *For every* $x_1 \in M_k^8$, $x_2 \in M$ *if*

$$\Delta_k(x_1;\ c_2, \gamma_1) \cap \Delta_k(x_2;\ c_2, \gamma_1) \neq \emptyset$$

*then* $x_1 = x_2$.

Proof. Let us fix $x \in \Delta_k(x_1;\ c_2, \gamma_1) \cap \Delta_k(x_2;\ c_2, \gamma_1)$. Note that $c_2 < \frac{1}{2}$, hence in view of (4) (with $\theta = \frac{1}{2}$), $\delta(x_2) \leqslant 2\delta(x) \leqslant 3\delta(x_1)$. In particular,

$$\|p(x_2) - p(x_1)\| \leqslant \|p(x_2) - p(x)\| + \|p(x) - p(x_2)\| <$$

$$< c_2[\delta^{\gamma_1}(x_2) + \delta^{\gamma_1}(x_1)] \leqslant c_2(3^{\gamma_1} + 1)\delta^{\gamma_1}(x_1) < c_1 \delta^{\gamma_1}(x_1).$$

Thus $x_2 \in \Delta_k(x_1;\ c_1, \gamma_1)$ and hence by Lemma 1, $x_1 = x_2$. The proof of Lemma 2is finished.

Define $U_k^j = \bigcup_{x \in M_k^j} \Delta_k(x;\ c_2, \gamma_1)$, $k = 1, \ldots, N$, $j = 1, \ldots, 8$.

In view of Lemma 2, mapping $\pi_k\colon U_k^8 \to M_k^8$ given by the relation:

$$\pi_k(y) = x \Leftrightarrow y \in \Delta_k(x;\ c_2, \gamma_1)$$

is a well-defined retraction such that $\pi_k(U_k^j) = M_k^j = M \cap U_k^j$, $j = 1, \ldots, 8$.

We pass to the study of properties of the coverings $(U_k^j)_{k=1}^N$ and retractions $(\pi_k)_{k=1}^N$ Let $c_3 = t^8(n^2 A 4^{\alpha+3})^{-1}$.

LEMMA 3 (comp. [3], Lemma 7). *For every* $x \in M_k^j$, $y \in \hat{B}(x, c_3\delta^{\gamma_1}(x))$:

$$\left|\frac{\partial G}{\partial x_k}(y)\right| > t^{j+1}\delta^{\beta}(y),\quad k = 1, \ldots, N,\ j = 1, \ldots, 8.$$

Proof. In view of (6), (4), for any $y \in \hat{B}(x, \tfrac{1}{2}\delta(x))$:

$$\left|\frac{\partial G}{\partial x_k}(y) - \frac{\partial G}{\partial x_k}(x)\right| \leqslant n^2 2^{2\alpha+5} A \delta^{-(\alpha+2)}(x) \| p(y) - p(x) \| .$$

In particular, for $y \in \hat{B}(x, c_3\delta^{\gamma_1}(x))$ we get:

$$\left|\frac{\partial G}{\partial x_k}(y) - \frac{\partial G}{\partial x_k}(x)\right| < \frac{1}{2} t^8 \delta^{\beta}(x) \leqslant \frac{1}{2} t^j \delta^{\beta}(x),$$

hence $\left|\frac{\partial G}{\partial x_k}(y)\right| > \frac{1}{2} t^j \delta^{\beta}(x) \geqslant t^{j+1}\delta^{\beta}(y)$. The proof of Lemma 3 is completed.

Let $q_k = (p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n)\colon X \to C^{n-1}$, $k = 1, \ldots, n$.
Define $c_4 = \frac{1}{4}\min\{c_3, c_2 2^{-(\gamma_1+1)}\}$, $c_5 = c_4 t^8 (n^{3/2} A 4^{\alpha+2})^{-1}$, $c_6 = t^{-9}\sqrt{n} A 2^{\alpha+1}$, $\gamma_2 = 2\alpha+2\beta+3$, $\gamma_3 = \alpha+\beta+1$.

LEMMA 4. *Let us fix* $x_0 \in M_k^8$ *and let* $Y = \{x \in X\colon\ |q_k(x) - q_k(x_0)| < c_5\delta^{\gamma_2}(x_0)\}$. *Then there exists a holomorphic mapping*

$$\Psi\colon\ Y \to M \cap \hat{B}(x_0, c_3\delta^{\gamma_1}(x_0))$$

*such that*

$$q_k \circ \Psi = q_k,$$

$$|p_k - p_k \circ \Psi| < c_4 \delta^{\gamma_1}(x_0),$$

$$\Psi(x) = x_0 \ \textit{provided that}\ q_k(x) = q_k(x_0),$$

$$\delta^{\gamma_3}(\Psi)\left|\frac{\partial(p_r \circ \Psi)}{\partial x_s}\right| \leqslant c_6,\quad r, s = 1, \ldots, n.$$

Proof. Let $G_0 = G \circ \chi$, where $\chi = (p|_{\hat{B}(x_0)})^{-1}$. Put

$$U = B(q_k(x_0), c_5\delta^{\gamma_2}(x_0)) \subset \mathbf{C}^{n-1}, \quad D = B(p_k(x_0), c_4\delta^{\gamma_1}(x_0)) \subset \mathbf{C}.$$

Observe that $U \times D \subset\subset p(\hat{B}(x_0, c_3\delta^{\gamma_1}(x_0)))$. By Lemma 3:

$$\left|\frac{\partial G_0}{\partial z_k}(z)\right| > t^9\delta^\beta(\chi(z)), \quad z \in U \times D. \tag{11}$$

In view of Lemma 1, the function

$$\bar{D} \in \lambda \to G_0(q_k(x_0), \lambda)$$

has exactly one zero $\lambda = p_k(x_0)$ and

$$|G_0(q_k(x_0), \lambda)| \geq \tfrac{1}{2}c_4 t^8 \delta^{\beta+\gamma_1}(x_0), \qquad \lambda \in \partial D.$$

On the other hand, using methods analogous to those used in the proof of Lemma 3 we get:

$$|G_0(w, \lambda) - G_0(q_k(x_0), \lambda)| \leq n^{3/2} A 4^{\alpha+1} \delta^{-(\alpha+1)}(x_0)|w - q_k(x_0)| \leq$$
$$\leq \tfrac{1}{4}c_4 t^8 \delta^{\beta+\gamma_1}(x_0), \quad w \in U, \ \lambda \in \partial D.$$

Hence, by the Rouché theorem, for every $w \in U$, the function

$$D \ni \lambda \to G_0(w, \lambda)$$

has exactly one zero $\lambda = \varphi(w)$. By the implicit function theorem, the function

$$\varphi\colon U \to D$$

is holomorphic and

$$\frac{\partial\varphi}{\partial w_s}(w) = -\frac{\partial G_0}{\partial z_s}(w, \varphi(w))\left[\frac{\partial G_0}{\partial z_k}(w, \varphi(w))\right]^{-1}, \quad w \in U, \ s \neq k.$$

In particular, in view of (11):

$$\delta^{\gamma_3}(\chi(w, \varphi(w)))\left|\frac{\partial\varphi}{\partial w_s}(w)\right| \leq c_6, \quad w \in U, \ s \neq k.$$

Now it is seen that we can put

$$\Psi(x) = \chi(q_k(x), \varphi(q_k(x))), \qquad x \in Y.$$

The proof of Lemma 4 is finished.

LEMMA 5 (*A characterization of the coverings* $(U_k^j)_{k=1}^N$ *and retractions* $(\pi_k)_{k=1}^N$).

(i) $U_k^j \in \operatorname{top} X$, $k = 1, \ldots, N$, $\quad j = 1, \ldots, 8$;

(ii) $\pi_k$ *is holomorphic and* $\delta^{\gamma_3}(\pi_k)\left|\dfrac{\partial(p_r \circ \pi_k)}{\partial x_s}\right| \leq c_6$, $k = 1, \ldots, N$, $\quad r, s = 1, \ldots, n$;

(iii) *For every* $x_0 \in M_k^j$: $\hat{B}(x_0, c_5\delta^{\gamma_2}(x_0)) \subset U_k^{j+1}$, $\quad k = 1, \ldots, N$, $\quad j = 1, \ldots, 7$;

(iv) $\bar{U}_k^j \subset U_k^{j+1}$, $\quad k = 1, \ldots, N$, $\quad j = 1, \ldots, 7$.

Proof. Let us fix $x_* \in U_k^j$. Put $x_0 = \pi_k(x_*)$ and let $Y, \Psi$ have the same meaning as in Lemma 4. Note that, in view of Lemma 3, $\Psi(Y) \subset M_k^{j+1}$. Put $B = \hat{B}(x_*, c_5\delta^{\gamma_2}(x_0))$ $(c_5\delta^{\gamma_2}(x_0) \leqslant c_5\delta(x_0) \leqslant 2c_5\delta(x_*) < \varrho(x_*))$. Observe that $B$, $\Psi(\hat{B}) \subset B(x_0, \frac{1}{3}\varrho(x_0))$, hence for every $x \in B$: $x \in \hat{B}(\Psi(x))$. In view of the continuity of $\Psi$, there exists $\varepsilon > 0$ such that:

$$\Psi(x) \in M_k^j,\ |p_k(x) - p_k(\Psi(x))| < \frac{j}{8} c_2\delta^{\gamma_1}(\Psi(x)),\ x \in \hat{B}(x_*, \varepsilon).$$

Thus $\hat{B}(x_*, \varepsilon) \subset U_k^j$ and $\pi_k = \Psi$ in $\hat{B}(x_*, \varepsilon)$ which, in view of Lemma 4, gives (i) and (ii).

If $x_* = x_0$ then for $x \in B$ we have

$$|p_k(x) - p_k(\Psi(x))| \leqslant |p_k(x) - p_k(x_0)| + |p_k(x_0) - p_k(\Psi(x))| <$$
$$< c_5\delta^{\gamma_2}(x_0) + c_4\delta^{\gamma_1}(x_0) \leqslant 2c_4\delta^{\gamma_1}(x_0) \leqslant 2^{\gamma_1+1}c_4\delta^{\gamma_1}(\Psi(x)) \leqslant \tfrac{1}{4}c_2\delta^{\gamma_1}(\Psi(x)) \leqslant$$
$$\leqslant \frac{j+1}{8} c_2\delta^{\gamma_1}(\Psi(x)).$$ Hence we get (iii).

For the proof of (iv), let $x = \lim_{m \to +\infty} x_m$ where $x_m \in \hat{B}(x) \cap \Delta_k\left(x_m^0, \frac{j}{8}, \gamma_1\right)$, $x_m^0 \in M_k^j$, $m \geqslant 1$. Observe that

$$\|p(x) - p(x_m^0)\| \leqslant \|p(x) - p(x_m)\| + \|p(x_m) - p(x_n^0)\| < \|p(x) - p(x_m)\| + c_2\delta(x_m^0) \leqslant$$
$$\leqslant \|p(x) - p(x_m)\| + 2c_2\delta(x_m),\ m \geqslant 1.$$

Hence there exists $m_0$ such that for $m \geqslant m_0$:

$$\|p(x) - p(x_m^0)\| \leqslant \tfrac{1}{8}\varrho(x) + 4c_2\delta(x) \leqslant \tfrac{1}{4}\varrho(x).$$

Since $\hat{B}(x) \cap B(x_m^0) \neq \emptyset$, this means that $x_m^0 \in \hat{B}(x, \frac{1}{4}\varrho(x))$, $m \geqslant m_0$. The last "ball" is relatively compact, so there exist a subsequence $(x_{m_l})_{l=1}^{\infty}$ and a point $x^0 \in \overline{\hat{B}(x, \frac{1}{4}\varrho(x))}$ such that $x^0 = \lim_{l \to +\infty} x_{m_l}$. Obviously $x^0 \in \overline{M}_k^j \subset M_k^{j+1}$, $q_k(x) = \lim_{l \to +\infty} q_k(x_{m_l}) = \lim_{l \to +\infty} q_k(x_{m_l}^0)$ $= q_k(x^0)$ and $|p_k(x) - p_k(x^0)| = \lim_{l \to +\infty} |p_k(x_{m_l}) - p_k(x_{m_l}^0)| \leqslant \frac{j}{8} c_2 \lim_{l \to +\infty} \delta^{\gamma_1}(x_{m_l}^0) = \frac{j}{8} c_2\delta^{\gamma_1}(x^0) <$ $< \frac{j+1}{8} c_2\delta^{\gamma_1}(x^0)$. Thus $x \in U_k^{j+1}(x^0 = \pi_k(x))$. The proof of Lemma 5 is finished.

Define $U_0 = X \setminus \bigcup_{k=1}^{N} \overline{U}_k^2$, $U_k = U_k^7$, $k = 1, \ldots, N$. We shall show that the covering $U_0, U_1, \ldots, U_N$ satisfies all the required conditions (comp. Prop. 1).

II. Partition of unity.

The method of the construction is taken from [3].

Let $\xi, \zeta \in C^\infty(\boldsymbol{C}, [0, 1])$ be such that:

$$\xi(z) = 1 \text{ if } |z| \leqslant \tfrac{3}{8}c_2, \quad \xi(z) = 0 \text{ if } |z| \geqslant \tfrac{4}{8}c_2,$$
$$\zeta(z) = 0 \text{ if } |z| \leqslant t^4, \quad \zeta(z) = 1 \text{ if } |z| \geqslant t^3,$$

and let $c_7 = c_7(\beta, c_2) = \max\left\{\left\|\frac{\partial \xi}{\partial z}\right\|_\infty, \left\|\frac{\partial \zeta}{\partial z}\right\|_\infty\right\}$.