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Differential Stability in Parametrized Nonlinear Programming

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Introduction. Stability and differential stability of a mathematical programming problem under perturbations have been studied extensively in the literature. Several authors investigated the continuity and the differential properties of the extremal value function as well as semicontinuity of the optimal set map for the case of nonlinear and nonconvex programs under perturbations (see [2, 4, 5, 7, 8, 11]). In the case when perturbations are linear with respect to parameter Gauvin and Tolle [7] have obtained the upper and lower bounds of the directional derivatives of the extremal value function under a rather mild assumption.

In this paper we shall extend the main results of Gauvin and Tolle [7] to the general case of parametrized nonlinear programs. Moreover, we shall calculate the upper and lower bounds of directional Dini derivatives of the extremal value function and, therefore, a sufficient condition for the continuity of the optimal set map will follow.

It is worth noticing that much stronger results are known but under more restrictive assumptions. For instance, in [1, 6] it was shown that the extremal value function is differentiable in a neighbourhood of a local optimal solution and its gradient is equal to the Kuhn-Tucker multiplier vector corresponding to that local optimal solution provided the second order sufficient condition and the strict complementarity condition hold.

Notation and preliminary results will be given in Section 1. In Section 2, under the assumption on regularity, the "local" stability of a feasible set will be established. Continuity of the extremal value function and semicontinuity of the optimal set map will be investigated in Section 3. The results obtained in two preceding sections will be used in Section 4 to estimate upper and lower bounds of the directional Dini derivatives of the extremal value function. Finally, we state a sufficient condition for the continuity of the optimal set map.

I. Preliminaries. In this paper we shall consider the following parametrized nonlinear program:

$$(P_a) \equiv \begin{cases} \text{minimize} & f(x, a) \\ \text{subject to} & g_i(x, a) < 0 \quad i = 1, \ldots, m \\ & h_j(x, a) = 0 \quad j = 1, \ldots, p \end{cases}$$

where $x \in R^n$, $a \in R^q$, a plays the role of *a parameter vector*.

The program

$$(P_0) \equiv \begin{cases} \text{minimize} & f(x,0) \\ \text{subject to} & g_i(x,0) \leqslant 0 \quad i = 1, \dots, m \\ & h_j(x,0) = 0 \quad j = 1, \dots, p \end{cases}$$

corresponding to the parameter $a = 0$ is considered an unperturbed program. Throughout this paper f, g_i, h_j are continuous functionals of two variables x and a.

Given a continuous functional $s\colon R^{n+q} \to R$ and a vector $\bar{a} \in R$ we define *directional Dini derivatives* of s:

$$\bar{D}_a s(x^0, a^0;\ \bar{a}) = \limsup_{\substack{a \to \bar{a} \\ \alpha \to 0^+}} \frac{s(x^0, a^0 + \alpha a) - s(x^0, a^0)}{\alpha}$$

$$\underline{D}_a s(x^0, a^0;\ \bar{a}) = \liminf_{\substack{a \to \bar{a} \\ \alpha \to 0^+}} \frac{s(x^0, a^0 + \alpha a) - s(x^0, a^0)}{\alpha}.$$

Assume further that

(a) h_j are continuously differentiable with respect to (x, a).

(b) f, g_i are differentiable with respect to the first variable x and the partial derivatives $\nabla_x f(x, a)$, $\nabla_x g_i(x, a)$ are continuous.

(c) $\bar{D}_a f(x^0, 0;\ a)$, $\underline{D}_a f(x^0, 0;\ a)$, $\bar{D}_a g_i(x^0, 0;\ a)$, $\underline{D}_a g_i(x^0, 0;\ a)$ are finite for all $a \in R^q$ at each feasible solution x^0 of the program (P_0).

For fixed parameter vector $a \in R^q$ the *feasible set* for (P_a) will be denoted by $S(a)$, that is

$$S(a) = \left\{ x \in R^n \middle| \begin{matrix} g_i(x, a) \leqslant 0 & i = 1, \dots, m \\ h_j(x, a) = 0 & j = 1, \dots, p \end{matrix} \right\}.$$

We set $D = \{a \in R^q | S(a) \neq \emptyset\}$. Assume that $0 \in D$. The *extremal value function* $f_{\inf}(a)$ is defined by

$$f_{\inf}(a) = \begin{cases} \inf\{f(x, a), x \in S(a)\} & \text{if } S(a) \neq \emptyset \\ +\infty & \text{if } S(a) = \emptyset. \end{cases}$$

For $a \in D$ we define the *optimal solution set* of problem (P_a) as follows

$$P(a) = \{x \in S(a) | f(x, a) = f_{\inf}(a)\}.$$

Sometimes we shall consider $S\colon a \to S(a)$ and $P\colon a \to P(a)$ as point-to-set maps from R^q into the subsets of R^n. For convenience of the reader we recall the definitions of some of concepts needed in the sequel.

Definitions [3, 7, 9]. Let Z and X be a two topological spaces and $F\colon Z \to X$ be a point-to-set map.

We say that F is *open* at $a^0 \in Z$ if for any sequence $\{a^k\}$ converging to a^0 and $x^0 \in F(a^0)$, a sequence $\{x^k\}$ and a number k_0 can be found such that $x^k \to x^0$ and $x^k \in F(a^k)$ for all $k \geqslant k_0$.