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Tangent Cones and Differential Stability of Parametrized Nonconvex Programming

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Introduction. Tangent cones play an important role in the theory of the mathematical programming. They have been used to deduce the most general necessary conditions for an optimal solution of the mathematical programming problem. In this paper we show that tangent cones are also useful in studying the differential stability of parametrized programming. Using tangent cones the upper and lower bounds of the directional Dini derivatives of the extremal value function (as a function of a parameter) will be obtained, provided that a regularity condition holds. The results in the present paper generalize the similar statements obtained by the author in [4].

Notations and preliminary results will be given in Section 1. The parametrized nonconvex programming of a rather general form and a regularity condition, which was given by Penot [7] in terms of tangent cones, will be presented in Section 2. Section 3 is devoted to the fundamental Lemma (Lemma 3.1) needed in the following sections and to the "local" stability of the feasible set for the mathematical programming problem. The main results, estimations of the upper and lower bounds of directional Dini derivatives of the extremal value function, will be given in Section 4. Finally, we shall give a comparison between two regularity conditions: $(R1_0)$ used in [4] and (R^r) used throughout this paper, and some conclusions.

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I. Preliminaries. We begin with definitions and results we need in the sequel.

1. **Tangent cones:** Let X be a topological vector space and B be a subset in X.

The tangent cone $T_{x_0}B$ to B at a point $x_0 \in \mathrm{cl}(B)$ is the set of vectors $v \in X$ such that there exists a sequence $(x_n, t_n) \in B \times (0, +\infty)$ with limit $(x_0, 0)$ and $v = \lim t_n^{-1}(x_n - x_0)$.

Some other equivalent definitions of the tangent cone can be found in [7].

The *radial tangent cone* $T_{x_0}^r B$ to B at a point $x_0 \in \mathrm{cl}(B)$ is the set of vectors $v \in X$ such that $x_0 + t_n v \in B$ for some sequence t_n in $(0, +\infty)$ with limit 0.

Obviously, $T_{x_0}^r B \subset T_{x_0} B$ and if B is starlike at x_0, in particular, if B is convex, then $T_{x_0}^r B = \mathrm{cone}\,(B - x_0)$ and $T_{x_0} B = \mathrm{cl}(T_{x_0}^r B)$.

The polar cone of $T_{x_0}B$,

$$(T_{x_0}B)^0 = \{x^* \in X' \colon \langle x^*, x\rangle \leqslant 0 \quad \text{for all } x \in T_{x_0}B\},$$

where X' is the topological dual of X, is called the *normal cone* to B at x^0 and is denoted by $N_{x_0}B$, i.e. $N_{x_0}B = (T_{x_0}B)^0$.

2. **Multivalued mappings.** Let F be a multivalued mapping from Z into X. We denote

$$\operatorname{dom} F = \{a \in Z \colon F(a) \neq \emptyset\},$$
$$\operatorname{graph} F = \{(x, a) \in X \times Z \colon x \in F(a)\}.$$

We say that F is *convex* (*closed*) if its graph is a convex (closed) set in $X \times Z$.

A multivalued mapping F from Z into X is said to be *surjective* if $F(Z) = X$. Let us notice that the condition $F(Z) = X$ is equivalent to $p_x(\operatorname{graph} F) = X$, where p_x is the projection $p_x \colon X \times Z \to X$.

We say that F is *locally surjective* at a point a_0 if F carries every neighborhood $\mathscr{U}$ of a_0 onto a neighborhood $F(\mathscr{U})$ of $F(a_0)$.

THEOREM 1.1. ([3, 10]): *Let X and Z be two Banach spaces and let $F \colon Z \to X$ be a closed convex multivalued mapping with the graph $G \subset X \times Z$. If for some $(b, a) \in G$ the projection $p_x \colon X \times Z \to X$ maps $T^r_{(b,a)}G$ onto X, that is:*

$$p_x(T^r_{(b,a)}G) = X,$$

then F is locally surjective at a.

Remark 1.2. The condition $p_x(T^r_{(b,a)}G) = X$ is equivalent to the condition which was used by Robinson [10]: that $b \in X$ is an *internal point* of the range of F and $a \in F^{-1}(b)$. Indeed, since G is convex, $T^r_{(b,a)}G = \operatorname{cone}(G-(b, a))$ and the condition $p_x(T^r_{(a,b)}G) = p_x(\operatorname{cone}(G-(b, a))) = X$ means that for any $x \in X$ there exists $z \in Z$ and $t>0$ such that $t(x, z)+(b, a) \in G$ or, equivalently, $b+tx \in F(a+tz)$. This is equivalent to b being an internal point of the range of F.

A multivalued mapping F is said to be *uniformly compact near* a_0 if there exists a neighborhood $\mathscr{V}$ of a_0 such that $\operatorname{cl}\big(\bigcup_{a \in \mathscr{V}} F(a)\big)$ is compact.

Assume that X and Z are two metric spaces. We say that $F \colon Z \to X$ is θ-Lipschitzian on $A \subset Z$ if $\varrho(F(a), F(a')) < \theta d(a, a')$, where $\varrho(A, B)$ denotes the Hausdorff distance between A and B.

The following lemma on multivalued contraction mappings, essentially due to Nadler [6], will be used.

LEMMA 1.3. *Let B be an open ball with center a_0 and radius r in a complete metric space X. Let $F \colon B \to X$ be a θ-Lipschitzian on B multivalued mapping with closed nonempty values, where $0<\theta<1$. If $d(a_0, F(a_0))<(1-\theta)r$, then F has a fixed point $\bar{x} \in B$, i.e. $\bar{x} \in F(\bar{x}) \cap B$.*

Moreover, for any $\delta>0$ one can find a fixed point $\bar{x}$ which satisfies:

$$d(x_0, \bar{x}) \leqslant \frac{1+\delta}{1-\theta} d(x_0, F(x_0)).$$

3. **Convex process.** A multivalued mapping F from Z into X is called a *convex process* if its graph is a convex cone in $X \times Z$, containing the origin.

Let X and Z be two normed spaces. A convex process F is said to be *bounded* if there is a number $c>0$ such that

$$\|F(a)\| = \sup_{x \in F(a)} \|x\| \leqslant c\|a\|,$$

for all $a \in \operatorname{dom} F$.

If F is a convex process, then the multivalued mapping F^* from X' into Z' defined by

$$F^*(x^*) = \{a^* \in Z' : (-x^*, a^*) \in (\operatorname{graph} F)^0\}$$

is also a convex process. It is called the *conjugate* of F.

PROPOSITION 1.4. *Let X be a reflexive Banach space, Z be a finite dimensional normed space and F be a closed convex process from Z into X. Let F^* be the conjugate of F. The following statements are equivalent to each other:*

(i) $\operatorname{dom} F = Z$

(ii) $F^*(0) = \{0\}$

(iii) *$F^*(x^*)$ is bounded for some $x^* \in \operatorname{dom} F^*$*

(iv) *$F^*(x^*)$ is bounded for every $x^* \in \operatorname{dom} F^*$*

(v) *F^* is bounded.*

Proof. It is clear from the definition that (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii). It suffices to prove that (i) is equivalent to (ii) and (ii) $\Rightarrow$ (v).

By definition, we have $\langle a^*, a\rangle \leqslant 0$ for all $a \in \operatorname{dom} F$ and $a^* \in F^*(0)$. Hence, $\operatorname{dom} F = Z$ implies $F^*(0) = \{0\}$. Conversely, if $\operatorname{dom} F \neq Z$, using theorem on separation of convex sets, we get $F^*(0) \neq \{0\}$. So (i) is equivalent to (ii).

Suppose that F is not bounded. It means that there exists a sequence (x_k^*, a_k^*) such that $a_k^* \in F^*(x_k^*)$ and $\dfrac{\|x_k^*\|}{\|a_k^*\|} \to 0$. Since F^* is a convex process, $\dfrac{a_k^*}{\|a_k^*\|} \in F^*\left(\dfrac{x_k}{\|a_k^*\|}\right)$. Subsequencing if necessary, we can assume that the sequence $\dfrac{a_k^*}{\|a_k^*\|}$ converges to a_0^*, $a_0^* \neq 0$. Since F^* is weak*-closed, $a_0^* \in F^*(0)$. This contradicts (ii). So (ii) implies (v).

PROPOSITION 1.5. *The assumptions are as in Proposition 1.4. If $\operatorname{dom} F = Z$, then for all $x_0^* \in \operatorname{dom} F^*$ we have*

$$\sup_{a^*}\{\langle a^*, \bar{a}\rangle : a^* \in F^*(x_0^*)\} = \inf_{\bar{x}}\{\langle x_0^*, \bar{x}\rangle : \bar{x} \in F(\bar{a})\}. \tag{1}$$

Proof. By the definition of F^*, we have $-\langle x^*, x\rangle + \langle a^*, a\rangle \leqslant 0$ for all $x \in F(a)$ and $a^* \in F^*(x^*)$. Therefore,

$$\sup_{a^*}\{\langle a^*, \bar{a}\rangle : a^* \in F^*(x_0^*)\} \leqslant \inf_{\bar{x}}\{\langle x_0^*, \bar{x}\rangle : \bar{x} \in F(\bar{a})\}. \tag{2}$$

Let us denote $G = \text{graph} F$ and $\gamma = \sup_{a^*}\{\langle a^*, \bar{a}\rangle : a^* \in F^*(x_0^*)\}$. By the assumption, γ is finite. Given an arbitrary number $\varepsilon > 0$, we shall show that there exists a convex neighborhood $\mathcal{U}$ of x_0^* in X' such that the set $\Gamma_\varepsilon = \{(-x^*, a^*) : x^* \in \mathcal{U}, \langle a^*, \bar{a}\rangle > \gamma+\varepsilon\}$ does not intersect G^0, that is $\Gamma_\varepsilon \cap G^0 = \emptyset$. If this were not the case, then we can find a sequence $\{(x_k^*, a_k^*)\}$ such that $x_k \to x_0$ and $a_k^* \in F^*(x_k^*)$ and $\langle a_k^*, \bar{a}\rangle > \gamma+\varepsilon_0$. The sequence $\{a_k^*\}$ is bounded. Indeed, if there is a subsequence $\{a_l^*\}$ with $\|a_l^*\| \to \infty$, then by the similar reasoning as in the proof of Proposition 1.4, we get $F^*(0) \neq \{0\}$, which contradicts the assumption that $\text{dom} F = Z$. Hence, without loss of generality, we can assume that the sequence $\{a_k^*\}$ converges to a_0^*. By the closedness of F^*, $a_0^* \in F^*(x_0^*)$. On the other hand, we have $\langle a_0^*, \bar{a}\rangle = \lim_{k\to\infty} \langle a_k^*, \bar{a}\rangle \geqslant \gamma+\varepsilon_0$ which is impossible.

Now, using the theorem on separation of two disjoint convex sets, we get the existence of $(x_0, a_0) \in X \times Z$, $(x_0, a_0) \neq 0$ and

$$\langle x^*, x_0\rangle + \langle a^*, a_0\rangle \geqslant 0 \quad \text{for all } (x^*, a^*) \in \Gamma_\varepsilon \tag{3}$$

$$\langle x^*, x_0\rangle + \langle a^*, a_0\rangle \leqslant 0 \quad \text{for all } (x^*, a^*) \in G^0. \tag{4}$$

Fixing $x^* \in \mathcal{U}$, we get from (3) that if $\langle a^*, \bar{a}\rangle \geqslant \gamma+\varepsilon$ then $\langle a^*, a_0\rangle \geqslant 0$.

Hence the following implications follows:

$$\langle a^*, a_0\rangle < 0 \Rightarrow \langle a^*, \bar{a}\rangle \leqslant 0 \quad \text{and} \quad \langle a^*, \bar{a}\rangle < 0 \Rightarrow \langle a^*, a_0\rangle \leqslant 0\,. \tag{5}$$

We shall show that (5) implies $a_0 = \lambda\bar{a}$, $\lambda > 0$.

Let us set $Q = \{a^* \in Z' : \langle a^*, a_0\rangle \leqslant 0\}$, $\hat{Q} = \{a^* \in Z' : \langle a^*, \bar{a}\rangle \leqslant 0\}$ and $\tilde{Q} = \{\lambda\bar{a} : \lambda \geqslant 0\}$. Then, by (5), $Q = \hat{Q}$. By definition $\tilde{Q}^0 = \{a^* \in Z' : \langle a^*, \bar{a}\rangle \leqslant 0\} = \hat{Q} = Q$. Since $\tilde{Q}$ is closed, $\tilde{Q}^{00} = \tilde{Q} = \hat{Q}^0 = Q^0$. Therefore, $a_0 = \lambda\bar{a}$, $\lambda \geqslant 0$. If $\lambda = 0$, then (3) implies that $x_0 = 0$, which contradicts the fact that $(x_0, a_0) \neq 0$. Hence, $\lambda \neq 0$. We can assume that $\lambda = 1$.

We get from (4):

$$\langle x^*, x_0\rangle + \langle a^*, \bar{a}\rangle \leqslant 0 \quad \text{for all } (x^*, a^*) \in G^0.$$

Since G is a closed convex cone, the above inequality is equivalent to the relation $(x_0, \bar{a}) \in G$, or $x_0 \in F(\bar{a})$. Substituting $x^* = -x_0^*$ in (3) we get $\langle x_0^*, x_0\rangle \leqslant \gamma+\varepsilon$.

We have proved that given an arbitrary number $\varepsilon > 0$ there exists $x_0 \in F(\bar{a})$ such that $\langle x_0^*, x_0\rangle \leqslant \gamma+\varepsilon$. This together with (2) gives the required equality. From the proof of Proposition 1.5 we get easily the following

COROLLARY 1.6. *The assumptions are as in Proposition* 1.4. *Given* $\bar{a} \in Z$, *let us set*

$$m(x^*) = \begin{cases} \sup_{a^*}\{\langle a^*, \bar{a}\rangle : a^* \in F^*(x^*)\} & \text{if } x^* \in \text{dom} F^* \\ -\infty & \text{if } x^* \notin \text{dom} F^* \end{cases}$$

If $\text{dom} F = Z$, *then the functional* $m(x^*)$ *is upper semicontinuous.*

Remark 1.7. Let $F : Z \to X$ be a convex process. If we set

$$F_*(x^*) = \{a^* \in Z' : (-x^*, a^*) \in -(\text{graph} F)^0\},$$

then F_* is also a convex process from X' into Z'. By definition, we have $\langle x^*, x\rangle - \langle a^*, a\rangle \leqslant 0$ for all $x \in F(a)$ and $a^* \in F_*(x^*)$. So

$$\inf_{a^*}\{\langle a^*, \bar{a}\rangle : a^* \in F_*(x^*)\} \geqslant \sup_{\bar{x}}\{\langle x^*, \bar{x}\rangle : \bar{x} \in F(\bar{a})\}.$$

By the similar reasoning as in the proof of Proposition 1.5, we can show, with the same assumptions, that if $\operatorname{dom} F = Z$, then

(6) $$\inf_{a^*}\{\langle a^*, \bar{a}\rangle : a^* \in F_*(x^*)\} = \sup_{\bar{x}}\{\langle x^*, \bar{x}\rangle : \bar{x} \in F(\bar{a}) \quad \text{for all } x^* \in \operatorname{dom} F_*$$

(7) the functional $x \to w(x^*)$, where

$$w(x^*) = \begin{cases} \inf\{\langle a^*, \bar{a}\rangle : a^* \in F_*(x^*)\} & \text{if } x^* \in \operatorname{dom} F_* \\ +\infty & \text{if } x^* \notin \operatorname{dom} F_* \end{cases}$$

is lower semicontinuous.

Remark 1.8. Consider the convex programming problem

(8) $$\{\text{minimize} \langle x_0^*, x\rangle, \text{ subject to } x \in F(\bar{a})\}.$$

The problem $\{\text{maximize} \langle a^*, \bar{a}\rangle$, subject to $a^* \in F^*(x_0^*)\}$ can be viewed as the dual one of (8). Hence, Proposition 1.5 and Corollary 1.6 give us a sufficient condition for having the duality relation.

II. Regularity condition and Kuhn-Tucker vectors. Let X and Y be two Banach spaces, B and C be two subsets in X and Y, respectively. Given the functions $f_0: X \to R$ and $g_0: X \to Y$, we consider the following mathematical programming problem:

($\mathscr{P}_0$) $$\begin{cases} \text{minimize} & f_0(x) \\ \text{subject to} & x \in B \quad \text{and} \quad g_0(x) \in C. \end{cases}$$

Let Z be a finite dimensional normed space and A be an open subset in Z and $0 \in A$. Assume that f, g are functions from $X \times A$ into R and Y, respectively, and $f(x, 0) = f_0(x)$, $g(x, 0) = g_0(x)$ for all $x \in X$.

We shall deal with the following perturbed problem of ($\mathscr{P}_0$)

($\mathscr{P}_a$) $$\begin{cases} \text{minimize} & f(x, a) \\ \text{subject to} & x \in B \quad \text{and} \quad g(x, a) \in C. \end{cases}$$

For a fixed parameter $a \in A$ the feasible set for ($\mathscr{P}_a$) will be denoted by $S(a)$, that is $S(a) = \{x \in B : g(x, a) \in C\}$. For $a = 0$ the set $S(0)$ is, of course, the feasible set for problem ($\mathscr{P}_0$): $S(0) = S_0 = \{x \in B : g_0(x) \in C\}$.

The extremal value function $f_{\inf}(a)$ is defined by

$$f_{\inf}(a) = \begin{cases} \inf\{f(x, a) : x \in S(a) & \text{if } S(a) \neq \emptyset \\ +\infty & \text{if } S(a) = \emptyset. \end{cases}$$

We define the solution set of problem $(\mathscr{P}_a)$ as follows

$$P(a) = \{x \in S(a) \colon f(x, a) = f_{\inf}(a)\}.$$

So $a \to S(a)$ and $a \to P(a)$ can be considered as two multivalued mappings from A into the subsets of X and they will be denoted by S and P, respectively.

Let x_0 be a feasible solution for problem $(\mathscr{P}_0)$ and g be differentiable at x_0. The following regularity condition was given by Penot (see [7]):

$$(R^r) \qquad g'_0(x_0)(T^r_{x_0}B) - T^r_{g_0(x_0)}C = Y, \quad \text{and}$$

$$(L) \qquad T_{x_0}S_0 = T_{x_0}B \cap g'_0(x_0)^{-1}(T_{g_0(x_0)}C).$$

He has proved that (R^r) implies (L), provided B and C are closed convex subsets in X and Y, respectively, and g is strictly differentiable at x_0. In the case when Y is finite dimensional, (R^r) can be replaced by

$$(R) \qquad g'_0(x_0)(T_{x_0}B) - T_{g_0(x_0)}C = Y.$$

Clearly, (R^r) is stronger than (R).

Using the regularity condition given above, Penot deduced the necessary condition for a local minimum of problem $(\mathscr{P}_0)$, which we summarize in following

PROPOSITION 2.1. *Let x_0 be a local minimum for problem $(\mathscr{P}_0)$. Assume that B and C are convex sets and that f_0, g_0 are differentiable at x_0. If the regularity condition (R) and (L) hold at x_0, then there exists a vector $y^* \in Y'$ satisfying*

$$(9) \qquad \begin{cases} y^* \in N_{g_0(x_0)}C, & \text{and} \\ 0 \in f'_0(x_0) + y^* \circ g'_0(x_0) + N_{x_0}B. \end{cases}$$

Moreover, the set $K(x_0)$ of all such vectors y^*,

$$(10) \qquad K(x_0) = \{y^* \in N_{g_0(x_0)}C \colon 0 \in f'_0(x_0) + y^* \circ g'_0(x_0) + N_{x_0}B\},$$

is weak-closed, convex and bounded.*

A vector y^* satisfying (9) is called a *Kuhn-Tucker vector* (or *Lagrange multiplier*) associated with the local minimum x_0.

Now we assume that g is differentiable with respect to (x, a) at $(x_0, 0)$ and we denote by $g'(x_0, 0)$, $g'_x(x_0, 0)$ and $g'_a(x_0, 0)$ the derivative and the partial derivatives of g with respect to x and a at $(x_0, 0)$, respectively. So, it is known ([1]) that

$$g'(x_0, 0) = g'_x(x_0, 0) \circ p_x + g'_a(x_0, 0) \circ p_a = g'_0(x_0) \circ p_x + g'_a(x_0, 0) \circ p_a,$$

where p_x and p_a are the projections from $X \times Z$ into X and Z, respectively.

Let us denote by G the graph of S, that is

$$G = \{(x, a) \colon a \in A,\ x \in B \text{ and } g(x, a) \in C\}.$$

We shall consider the following conditions which are motivated by (R^r) and (L):

$$(RG^r) \qquad g'(x_0, 0)\big(T^r_{(x_0,0)}(B\times A)\big) - T^r_{g(x_0,0)}C = Y$$

$$(LG) \qquad T_{(x_0,0)}G = T_{(x_0,0)}(B\times A) \cap g'(x_0, 0)^{-1}(T_{g(x_0,0)}C).$$

Since A is open and $0 \in A$, $T^r_{(x_0,0)}(B\times A) = T_{x_0}B\times Z$. So (R^r) implies (RG^r). Moreover, assuming that B and C are closed convex sets and g is strictly differentiable at $(x_0, 0)$, we get the following implications

$$(R^r) \Rightarrow (RG^R) \Rightarrow (LG).$$

The following lemma will be useful.

LEMMA 2.2. *Let x_0 be a local minimum for problem $(\mathcal{P}_0)$. Assume that B and C are closed convex sets and that the function f is differentiable, the function g is strictly-differentiable at $(x_0, 0)$. If (RG^r) is satisfied, then*

(11) $\quad \{a^* \in Z' : (-f'_x(x_0, 0), a^*) \in [T_{(x_0,0)}G]^0\} = \{a^* \in Z' :\ a^* = y^* \circ g'_a(x_0, 0),\ y^* \in K(x_0)\},$

where $K(x_0)$ is the set of Kuhn-Tucker vectors corresponding to x_0.

Proof. By the assumptions, we get (LG), that is

$$T_{(x_0,0)}G = T_{(x_0,0)}(B\times A) \cap g'(x_0, 0)^{-1}(T_{g(x_0,0)}C).$$

Since the tangent cone are closed and convex and the condition (RG^r) holds, we have (see [7]):

$$[T_{(x_0,0)}G]^0 = [T_{(x_0,0)}(B\times A)]^0 + g'(x_0, 0)^*[N_{g(x_0,0)}C],$$

where $g'(x_0, 0)^*$ is the conjugate of $g'(x_0, 0)$. Hence,

$$[T_{(x_0,0)}G]^0 = N_{x_0}B\times\{0\} + g'(x_0, 0)^*[N_{g(x_0,0)}C].$$

This means that $(-f'_x(x_0, 0), a^*) \in [T_{(x_0,0)}G]^0$ if and only if there exists $x^* \in N_{x_0}B$ and $y^* \in N_{g(x_0,0)}C$ such that $(-f'_x(x_0, 0) = x^* + y^*_0 \circ g'_x(x_0, 0)$ and $a^* = y^* \circ g'_a(x_0, 0)$. The proof of Lemma is completed.

III. Stability of the feasible set. We shall show that if the regularity condition (R^r) holds at $x_0 \in S(0)$, then the feasible set $S(0)$ is *stable* at x_0 in the sense that for any sequence $\{a_n\}$ converging to 0 a sequence $\{x_n\}$ can be found such that $x_n \in S(a_n)$ for sufficiently large n and $\{x_n\}$ converges to x_0 as $n\to\infty$.

The following Lemma, which will play an important role in establishing our results, is essentially due to Penot [7]. We only adopt it to the case when the parameter a is taken into account.

LEMMA 3.1. *Let B and C be closed convex subsets of X and Y, respectively. Let $u\colon B\to Y$ be a function, $x_0 \in B$ and $u(x_0) \in C$. Assume that*

(a) $\quad$ *$u(B) - C$ is absorbing in Y*