UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXV

1985

The Rate of Convergence of Iterates of the Frobenius-Perron Operator for Lasota-Yorke Transformations

by Marian JABŁOŃSKI, Zbigniew S. KOWALSKI and Jan MALCZAK

1. Introduction. In this note we estimate the rate of convergence in supremum norm of iterates of the Frobenius-Perron operator for the Lasota-Yorke type transformations. We shall show that this convergence is geometrical.

2. Convergence theorem. Denote by $(L_1, \| \|_1)$ the space of all functions defined on $[0, 1]$ for which $|f|$ is integrable, and by m the Lebesgue measure on $[0, 1]$. Let $T: [0, 1] \to \to [0, 1]$ be a measurable nonsingular function, i.e., if A is measurable, $m(A) = 0$ implies $m(T^{-1}(A)) = 0$. Given T, the Frobenius-Perron operator $P_T: L_1 \to L_1$ is given by the formula

$$P_T f(x) = \frac{d}{dx} \int\limits_{T^{-1}([0,x])} f(s)\,ds.$$

The operator P_T is linear, continuous and satisfies the following conditions:

(a) P_T is positive: $f \geqslant 0 \Rightarrow P_T f \geqslant 0$,

(b) P_T preserves integral

$$\int_0^1 P_T f dm = \int_0^1 f dm, \ f \in L_1,$$

(c) $P_{T^n} = P_T^n$ where T^n denotes the n-th iterate of T,

(d) $P_T f = f$ iff the measure $d\mu = f dm$ is invariant under T, i.e., $\mu(T^{-1}(A)) = \mu(A)$ for each measurable A,

(e) If A is invariant, i.e., $m(A \Delta T^{-1}(A)) = 0$, then, for $f \in L_1$ such that $m(\mathrm{supp} f \setminus A) = 0$, $m(\mathrm{supp} P_T f \setminus A) = 0$.

Denote by $D = D([0, 1], \Sigma, m)$ the set of all nonnegative functions $f \in L_1$ with $\|f\|_1 = 1$.

Let T be a given transformation of the unit interval $[0, 1]$ into itself satisfying the following conditions:

(i) there exists a partition $0 = a_0 < \dots < a_r = 1$ such that for each i $(i = 1, \dots, r)$ the restriction T_i of T to the open interval $(a_{i-1}\ a_i)$ is a C^1 function which can be extended to the closed interval $[a_{i-1}\ a_i]$ as a C^1 function,

(ii) $\varphi_i' = (T_i^{-1})'$ is of bounded variation over [0, 1] ($i = 1, \ldots, r$),
(iii) $\inf_{x\in[0,1]}|T'(x)| > 1$.
The following results are contained in [3, 5, 6, 7].

THEOREM 1. *If T: [0, 1] → [0, 1] satisfying conditions* (i)—(iii) *has an absolutely continuous measure μ invariant under T with support equal to* [0, 1], *then*

(1) *there exists a finite family* $\{g_i\}_{n=1}^{p} \subset D$ *such that* $P_T g_i = g_i$ ($i = 1, \ldots, p$),
(2) $D_i = \operatorname{supp} g_i$ *is a finite sum of intervals* ($i = 1, \ldots, p$),
(3) D_i *is invariant under* T ($i = 1, \ldots, p$),
(4) D_i *are disjoint* $\big(m(D_i \cap D_j) = 0$ for $i \neq j\big)$ *and*

$$m\Big([0, 1] \setminus \bigcup_{i=1}^{p} D_i\Big) = 0,$$

(5) *there exists such k that for each* $f \in L_1$

$$\lim_{n\to\infty} \|P_\varphi^n(f 1_{D_i}) - g_i \int_{D_i} f\,dm\|_1 = 0$$

and consequently

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} P_T^i f = \lim_{n\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} P_T^i P_\varphi^n f, \quad \text{where} \quad \varphi = T^k$$

(6) *there exists such q that for* $\varphi_1 = T^q$ *and for each* $f \in L_1$ *of bounded variation* $(\bigvee_0^1 f < \infty)$

$$\bigvee_0^1 P_{\varphi_1} f \leqslant s_1 \bigvee_0^1 f + M\|f\|_1$$

for some $s_1 \in (0, 1)$ *and* $M > 0$.

The main result is the following

THEOREM 2. *Let T satisfies the assumptions of Theorem* 1. *Then there exist constants* $K > 0$, $c > 0$ *and* $s \in (0, 1)$ *such that for all functions f of bounded variation over* [0, 1]

$$|P_\varphi^n f - \sum_{i=1}^{p} g_i \int_{D_i} f\,dm| \leqslant s^n K\Big(\bigvee_0^1 f + c\|f\|_1\Big)$$

where φ is as in (5).

Denote by W_i the space of all integrable functions of bounded variation over [0, 1] such that $\operatorname{supp} f \subset D_i$ and $\int f\,dm = 0$. In W_i we introduce the norm $\|f\|_{W_i} = \|f\|_1 + \bigvee_0^1 f$.

In order to prove Theorem 2 we need the following

LEMMA. *Let φ be from* (5). *Then,* $\|P_\varphi^n f\|_1 \to 0$ *as* $n \to \infty$ *uniformly in the unit ball in* W_i ($i = 1, \ldots, p$).

Proof. Assume it is not. Then, there exists i and $\varepsilon>0$ such that for any $n\in N$ there exists $k(n)>n$ and f_n from the unit ball in W_i which satisfy $\|P^{k(n)}f_n\|_1>\varepsilon$. By virtue of Helly's theorem the set $\{f_n\}$ is relatively compact in W_i. Therefore, there exists a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ which is convergent to f_0 belonging to the unit ball in W_i. By this, without loss of generality we may assume that the sequence $\{f_n\}$ is convergent to f_0 in L_1. Now, we have.

$$\begin{aligned}\|P^{k(n)}f_0\|_1 &= \|P^{k(n)}f_0-P^{kn}f_n+P^{k(n)}f_n\|_1\\ &\geqslant \|P^{k(n)}f_n\|_1-\|P^{k(n)}(f_0-f_n)\|_1\geqslant \varepsilon-\|P^{k(n)}(f_0-f_n)\|_1 .\end{aligned}$$

Since $\|P_\varphi^{k(n)}(f_0-f_n)\|_1\leqslant\|f_0-f_n\|_1\to 0$ we obtain $\|P_\varphi^{k(n)}f_0\|_1\geqslant\frac{\varepsilon}{2}$ for sufficiently large n. On the other hand, since $f_0=f_0^+-f_0^-$ ($(z)^+=\max\{0,z\}$ and $(z)^-=\max\{0,-z\}$) and $\int f_0^+=\int f_0^-$, by (5) we have $P^nf_0=P^nf_0^+-P^nf_0^-\to 0$ as $n\to\infty$ strongly in L_1. This contradiction completes the proof of the lemma.

Proof of Theorem 2. Let us fix $i\in\{1,\dots,p\}$ and put $W=W_i$. By (6) there exists constant M_1 such that for each f belonging to the unit ball in W and each $n\in N$

$$\bigvee_0^1 P_{\varphi_1}^n f\leqslant M_1 \tag{7}$$

and, moreover, there exists constant M_2 such that for each f of bounded variation

$$\bigvee_0^1 P_{\varphi_1}^n f\leqslant s_1^j\bigvee_0^1 P_{\varphi_1}^{n-j}f+M_2\|P_{\varphi_1}^{n-j}f\|_1 \tag{8}$$

(for details see [3, 6]). Setting $\psi=T^{kq}$ by (7) and (8) we have

$$\bigvee_0^1 P_\psi^n f\leqslant M_1 \tag{9}$$

for each f belonging to the unit ball in W, $n=1,2,\dots$ and

$$\bigvee_0^1 P_\psi^n f\leqslant s_2^j\bigvee_0^1 P_\psi^{n-j}f+M_2\|P_\psi^{n-j}f\|_1 \tag{10}$$

for each f of bounded variation, where $s_2=s_1^k$. By Lemma and (9) there exists such $\bar n$ that

$$s_2^{\bar n}M_1<\tfrac14 \tag{11}$$

and

$$M_2\|P_\psi^{\bar n}f\|_1<\tfrac14,\quad \|P_\psi^{\bar n}f\|_1<\tfrac14 \tag{12}$$

for each f from the unit ball in W and $n\geqslant\bar n$. By (10), (11) and (12) we have

$$\bigvee_0^1 P_\psi^{2\bar n}f+\|P_\psi^{2\bar n}f\|_1\leqslant s_2^{\bar n}\bigvee_0^1 P_\psi^{\bar n}f+M_2\|P_\psi^{\bar n}f\|_1+\|P_\psi^{2\bar n}f\|_1<\tfrac34, \tag{13}$$

whenever f belongs to the unit ball in W.

Now, let us consider P_ψ as the operator acting from the space W to the space W (it is possible by (3) and (e)) By (13) we have

(14) $$\|P_\psi^{2\bar{n}}\|_W \leqslant \tfrac{3}{4}\,.$$

Therefore, by (c) and (14), there exist constants $\bar{K}$ and $s \in (0, 1)$ such that

(15) $$\|P_\varphi^n\|_W \leqslant \bar{K}s^n.$$

It is obvious that for f such that $\int f\,dm = 0$

(16) $$|f| \leqslant \bigvee_0^1 f.$$

From the definition of the norm in W of the operator P_φ and by (15) and (16), for $f \in W$ we have

(17) $$|P_\varphi^n f| \leqslant \bar{K}s^n\Big(\bigvee_0^1 f + \|f\|_1\Big)\,.$$

Now, let $f \geqslant 0$ be of bounded variation. By (2) $f1_{D_i}$ is of bounded variation. Therefore $f1_{D_i} - \|f1_{D_i}\|_1 g_i \in W$. This and (17) gives us

$$|P_\varphi^n(f1_{D_i}) - \|f1_{D_i}\|_1 g_i| \leqslant \tilde{K}s^n\Big(\bigvee_0^1 f + c\|f\|_1\Big)$$

for some $\tilde{K}$ and c. Hence, since P_φ is linear and $f = f^+ - f^-$, $\bigvee_0^1 f = \bigvee_0^1 f^+ + \bigvee_0^1 f^-$ we have

$$|P_\varphi^n(f1_{D_i}) - g_i \cdot \int_{D_i} f| \leqslant \tilde{K}s^n\Big(\bigvee_0^1 f + c\|f\|_1\Big)$$

whenever f is of bounded variation. Since D_i are disjoint, the last inequality gives us the thesis of Theorem 2.

From Theorem 2 and (5) we have the following

COROLLARY. *If T satisfies the assumptions of Theorem* 2, *then there exist constants* $K > 0$ *and* $c > 0$ *such that for each f of bounded variation*

$$\left|\frac{1}{n}\sum_{i=0}^{n-1} P_T^i f - \sum_{i=1}^{p} g_i \int_{D_i} f\right| \leqslant \frac{K}{n}\Big(\bigvee_0^1 f + c\|f\|_1\Big)\,.$$

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Received April 25, 1981