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On Certain Theorem on Branched Coverings of the Linear Variety

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Abstract. The following theorem is proved:

Let X be a normal analitic subvariety of C^n and $\varphi: X \to C^d$ a branched covering with the sheet number $s(\varphi)$. If the set $\{y \in C^d: \#\varphi^{-1}(y) < s(\varphi)\}$ is an affine algebraic subvariety of C^d then X is almost-algebraic.

In [1] the following fact is proved.

THEOREM. *Let X be a d-dimensional almost-algebraic set (this means that there exists a biholomorphic map from X onto some affine algebraic variety), then there exists a branched covering $\varphi: X \to C^d$.*

Let X be an irreducible d-dimensional analytic set in C^n and $\varphi: X \to C^d$ be a branched covering. The question arises whether X is almost algebraic. The answer is "yes" if the set $\{y \in C^d: \varphi^{-1}(y) < s(\varphi)\}$ is affine algebraic.

The proof is based on the Grauert-Remmert theorem on extension of a branched covering ([4], [3]), on the completization of an affine algebraic variety and some characterization of affine varieties given by Goodman and Landman [2].

First we state some necessary definitions

Definition 1. Let X and Y be complex spaces. A holomorphic map $\varphi: X \to Y$ is said to be a *finite covering* if and only if φ is a branched and locally biholomorphic covering.

Definition 2. Let U and X be complex spaces, and let $\beta: U \to X$ be a holomorphic map. The mapping β is said to be an *extension of the space* U if the set $\beta(U)$ is open in X and β is a biholomorphic map onto $\beta(U)$.

Definition 3. Let U and X be complex spaces, Y a complex subspace of X and let $\varphi: U' \to U: = X \setminus Y$ be a branched covering. A branched covering $\varphi': X' \to X$ is said to be an *extension of the branched covering* φ if there exists an extension $\beta: U' \to X'$ of the space U' such that the diagram

$$\begin{tikzcd}
U' \arrow[r, "\beta"] \arrow[d, "\varphi"'] & X' \arrow[d, "\varphi'"] \\
U \arrow[r, "\alpha"] & X
\end{tikzcd}$$

commutes, where α denotes the natural inclusion.

Our main result is the following:

THEOREM 1. *Let U' be a normal analytic subvariety of $\mathbf{C}^n$ and let $\varphi\colon U' \to \mathbf{C}^r$ be a branched covering with the sheet number $s(\varphi)$. Suppose that the set $C(\varphi) := \{y \in \mathbf{C}^r\colon \#\varphi^{-1}(y) < s(\varphi)\}$ is an affine algebraic subvariety of $\mathbf{C}^r$. Then U' is almost-algebraic.*

Proof. We start from the following:

THEOREM of Grauert of Remmert ([4], [3]). *Let X be a normal complex space and let Y be an analytic subset of X such that the set $U := X \setminus Y$ is dense in X. Let U' be a normal complex space and let $\varphi\colon U' \to U$ be a branched covering. Let S be a thin analytic subset of X such that the map $\varphi|\varphi^{-1}(U \setminus U \cap S)\colon \varphi^{-1}(U \setminus U \cap S) \to U \setminus U \cap S$ is a finite covering. Then there exist a normal irreducible complex space X' which is an extension of the space U' and a branched covering $\varphi'\colon X' \to X$ which is an extension of the branched covering $\varphi\colon U' \to U$. Moreover, X' and φ' are uniquely determined up to isomorphism.*

Put $X := \mathbf{P}^r$ and let Y be the hyperplane at infinity relative to $U := \mathbf{C}^r$. Then $\varphi\colon U' \to U$ is a branched covering whence $\varphi|U' \setminus \varphi^{-1}(C(\varphi))\colon U' \setminus \varphi^{-1}(C(\varphi)) \to U \setminus C(\varphi)$ is a finite covering. Moreover, $C(\varphi)$ is a thin affine subset of U. Let S be a projective completion of $C(\varphi)$ in $X = \mathbf{P}^r$. Then S is a thin analytic set in X and assumptions of the Grauert-Remmert theorem are satisfied.

Hence there exist β, X' and φ' such that the diagram

$$\begin{tikzcd}
U' \arrow[r, "\beta"] \arrow[d, "\varphi"'] & X' \arrow[d, "\varphi'"] \\
U = \mathbf{C}^r \subset \arrow[r] & X = \mathbf{P}^r
\end{tikzcd}$$

commutes. Now we recall the Theorems R and S which will be used in our proof.

THEOREM R ([3]). *Let X be a complex space and let $\varphi'\colon X' \to \mathbf{P}^r$ be a branched covering. Then there exists a closed holomorphic embedding $\mu\colon X' \to \mathbf{P}^m$, where m is a natural number.*

Observe that well-known Chow's theorem implies that $\mu(X')$ is a projective subset of $\mathbf{P}^m$.

THEOREM S ([6]). *Let M and N be projective subsets of $\boldsymbol{P}^k$ and $\boldsymbol{P}^s$, respectively, and let $\tau\colon M^{an} \to N^{an}$ be a holomorphic map, where M^{an} means a complex space associate with the algebraic space M. Then τ is a regular map.*

Theorem R implies that in our situation the following diagram

$$\begin{tikzcd}
U^r \arrow[r, dashed, "\beta"] \arrow[d, "\varphi"'] & X' \arrow[r, shift left, "\mu"] \arrow[d, "\varphi'"] & \mu(X')\subset \boldsymbol{P}^m \arrow[l, shift left, "\mu^{-1}"] \\
\boldsymbol{C}^r \arrow[r] & \boldsymbol{P}^r &
\end{tikzcd}$$

commutes. Hence $\mu(X')$ is a normal projective subvariety of $\boldsymbol{P}^m$. The map $\tau := \varphi' \circ \mu^{-1}\colon \mu(X') \to \boldsymbol{P}^r$ is a regular branched covering by Theorem S.

We put $\alpha' := \mu \circ \beta$, $M := \mu(X')$ and let $\alpha\colon \boldsymbol{C}^r \to \boldsymbol{C}^r \cup \boldsymbol{P}^{r-1}$ be an inclusion map, where $\boldsymbol{P}^{r-1}$ denotes a hyperplane at infinity. Then the diagram

$$\begin{tikzcd}
U' \arrow[r, dashed, "\alpha'"] \arrow[d, "\varphi"] & M \arrow[d, "\tau"] \\
\boldsymbol{C}^r \arrow[r, "\alpha"] & \boldsymbol{P}^r = \boldsymbol{C}^r \cup \boldsymbol{P}^{r-1}
\end{tikzcd} \tag{$*$}$$

commutes. Observe that α' is a biholomorphic map from U' onto some open set $W := \alpha'(U')\subset M$. It follows from commutativity of the diagram ($*$) that $\tau^{-1}(\boldsymbol{C}^r) = W$. Since τ is a regular map, it is continuous in the Zariski topology on $\boldsymbol{P}^r$ and M. Hence W is a Zariski open subset of M, $\psi := \tau|W\colon W \to \boldsymbol{C}^r$ is a regular covering and W is a quasi-projective variety.

Now we will use the following

THEOREM on characterization of affine varieties [2]. *Let W be a quasi-projective variety. Then the following conditions are equivalent:*

1. *There exists a regular and algebraically proper map $\varphi\colon W \to T$, where T is an affine variety and W contains no 1-dimensional projective subvarieties.*
2. *W is an affine variety.*

Since there exists a biholomorphic map from W^{an} onto some analytic subset of $\boldsymbol{C}^n$, W^{an} is a Stein space. Let C be a projective subvariety of W. Then C^{an} is compact irreducible analytic subset of W^{an} and this implies that C is a point. Now it suffices to show that the map $\psi\colon W \to \boldsymbol{C}^r$ is algebraically proper. In order to prove this we consider an arbitrary quasi projective variety Y and the mapping $\psi \times 1_Y\colon W\times Y \ni (x, y) \to (\psi(x), y) \in \boldsymbol{C}^r \times Y$. It suffices to prove that this map is closed in the sense of Zariski. Observe that the map $\psi^{an}\colon W^{an} \to \boldsymbol{C}^r$ is proper. Then the map $(\psi \times 1_Y)^{an}\colon (W\times Y)^{an} \to (\boldsymbol{C}^r \times Y)^{an}$ is closed. Let us put $M := W\times Y$ and $f := \psi \times 1_Y$. Since the Zariski closure of $f(M)$ and its closure in the natural topology are the same (see [6], p. 12), our theorem is proved.

Remark. After acceptation by the publisher, Professor J. Noguchi called the author's attention to the fact that the Theorem is the same as Lemma 5.1 in his work "Meromorphic mappings of a covering space over C^n..." Hir. Math., vol. 6,2, 1976.

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