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The Rate of Convergence of Iterates of the Frobenius-Perron Operator for Transformations on the Real Line

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1. Introduction. In a joint paper by A. Lasota and J. Yorke [1], the authors gave a criterion which play an interesting and important role in analyzing the convergence of the sequence $P^n f$ of iterates of the Frobenius-Perron operator in L_1 norm. In particular, it is easy to prove with the use of this criterion that the sequence $P_T^n f$ is convergent in L_1 norm for a family of mappings of the real line into itself. This family contains as a special case the functions of the form $T(x) = a(\mathrm{tg}(bx+c))$. In this paper we show that the convergence $P_T^n f$ of iterates of the Frobenius-Perron operator corresponding to the point transformation T under consideration is geometrical in the L_1 norm and in the metric of uniform convergence if an integrable function f is of bounded variation on R.

2. Convergence theorem. Denote by $(L_1, \|\cdot\|_1)$ the space of all functions f defined on the real line for which $|f|$ is integrable, and by m the Lebesgue measure on R. Let $T\colon R \to R$ be a measurable nonsingular function, i.e., if A is measurable, $m(A) = 0$ implies $m(T^{-1}(A)) = 0$. Given T, the Frobenius-Perron operator $P_T\colon L_1 \to L_1$ is given by the formula:

$$P_T f(x) = \frac{d}{dx} \int\limits_{T^{-1}(-\infty, x]} f(s)\,ds .$$

The operator P_T is linear, continuous and satisfies the following conditions:

(a) P_T is positive: $f \geqslant 0 \Rightarrow P_T f \geqslant 0$;

(b) P_T preserves integrals:

$$\int\limits_{-\infty}^{+\infty} P_T f dm = \int\limits_{-\infty}^{+\infty} f dm, \; f \in L_1 ;$$

(c) $P_{T^n} = P_T^n$ where T^n denotes the n-th iterate of T;

(d) $P_T f = f$ iff the measure $d\mu = f dm$ is invariant under T, i.e., $\mu(T^{-1}(A)) = \mu(A)$ for each measurable A.

Let $\{I_k\}_{k=-\infty}^{+\infty}$ be a countable partition of the real line R such that

(i) each I_k is an open interval

(ii) $I_k \cap I_j = \emptyset$ for $k \neq j$

(iii) $R - \bigcup_{k=-\infty}^{\infty} I_k$ is a countable set

(iv) $\sup|I_k| = L < \infty$, where $|I_k|$ denotes the length of I_k.

In our paper we will consider a family of mappings $T\colon \bigcup I_k \to R$ satisfying the following conditions:

(v) for any k the restriction T_k of T to the interval I_k is differentiable and its derivative T_k is locally Lipschitzean

(vi) $|T_k'(x)| \geqslant q > 1$ for $x \in I_k$

(vii) $T_k(I_k) = R$ for each k

(viii) $\dfrac{|T_k''(x)|}{(T_k'(x))^2} \leqslant M < \infty$

(ix) $\omega(x) = \sup\limits_k \dfrac{|\psi_k'(x)|}{|I_k|}$ is integrable on R where $\psi_k = T_k^{-1}$.

It is shown in [4] that for a function T satisfying (i)—(ix) there exists a unique finite, absolutely continuous measure on R, invariant under T.

We shall show the following convergence theorem

THEOREM. *Let $T\colon \bigcup I_k \to R$ be a function satisfying* (i)—(ix). *Then there exist constants $K>0$, $C>0$ and $0<s<1$ such that for every nonnegative integrable function f with bounded variation on R we have*

$$|P_T^n f - g_\mu \|f\|_1| \leqslant s^n K\Big(\bigvee_{-\infty}^{+\infty} f + C\|f\|_1\Big) \tag{1}$$

and

$$\|P_T^n f - g_\mu \|f\|_1\|_1 \leqslant s^n K\Big(\bigvee_{-\infty}^{+\infty} f + C\|f\|_1\Big), \tag{2}$$

where g_μ is the density of the probabilistic measure μ invariant under T and $\bigvee_{-\infty}^{+\infty} f$ denotes the variation of f over R.

Denote by $D = D(R, m)$ the set of all densities, that is, all $f \in L_1$ such that $f \geqslant 0$ and $\|f\|_1 = 1$.

For all nonnegative Lipschitzean function in $C(R)$ ($C(R)$ denotes the space of all continuous functions on R) we define "regularity" of f to be

$$\mathrm{Reg} f = \sup\left\{\left|\frac{f'(x)}{f(x)}\right| : x \in R,\ f'(x) \text{ is defined and } f(x) > 0\right\}.$$

Define

$$H = \{f \in C(R)\colon f \geqslant 0,\ f \text{ is Lipschitzean, } \mathrm{Reg} f < \infty \text{ and } \|f\|_1 = 1\}.$$

It is easy to verify that the set H is dense in D in the norm $\|\cdot\|_1$.
In order to prove our Theorem we need the following three lemmas.

LEMMA 1. *Let T be as in Theorem. Then for any $f \in D$ the sequence $P_T^n f$ is convergent in $\|\cdot\|_1$ to the function g_μ and consequently*

$$\|P_T^n f\|_1 \to 0 \text{ if } n \to \infty \quad \textit{whenever} \quad \int f\,dm = 0\,. \tag{3}$$

Proof. Using the results of [1] it is sufficient to show that there is $h \in L_1$, $h \geqslant 0$ and $\|h\|_1 > 0$ such that

$$\|(P_T^n f - h)^-\|_1 \to 0 \quad \text{if } n \to \infty, \qquad \text{for } f \in H;$$

where $(z)^- = \max(0, -z)$.

In view of the proof of Theorem 1 of [4] we have the following estimations

$$\bigvee_{-\infty}^{+\infty} P_T^n f \leqslant q^{-n} \bigvee_{-\infty}^{+\infty} f + \frac{qM}{q-1}\|f\|_1 \tag{4}$$

and

$$|P_T^n f(x)| \leqslant \omega(x)\left(Lq^{n-1} \bigvee_{-\infty}^{+\infty} f + \left(1 + \frac{qM}{q-1}\|f\|_1\right)\right), \tag{5}$$

where q is from (vi), M is from (viii) and ω is from (ix). Therefore by (5) there exists a constant K_1 such that for any f of bounded variation with $\|f\|_1 = 1$

$$|P_T^n f(x)| \leqslant K_1 \omega(x) \text{ for large enough } n\,. \tag{6}$$

Using the same method as in the proof of Theorem 1 of [3] one can show that there exists a constant ϱ such that for any $f \in H$

$$\mathrm{Reg} P_T^n f < \varrho \quad \text{for } n \text{ large enough}\,. \tag{7}$$

Since $\|P_T^n f\|_1 = 1$ if $f \geqslant 0$ and $\|f\|_1 = 1$, from (6) it follows that there exists an interval $[A, B]$ such that for any $f \in D$ with bounded variation

$$\int_A^B P_T^n f\,dm > \tfrac{1}{2} \quad \text{for sufficiently large } n\,.$$

Thus, for any sufficiently large n there exists $x \in [A, B]$ such that

$$P_T^n f(x) \geqslant \frac{1}{2(B-A)}\,.$$

Hence, by (7) for $f \in H$ we obtain

$$P_T^n f(x) \geqslant \frac{1}{2(B-A)} \exp(-\varrho(B-A)) \quad \text{for } x \in [A, B]$$

and sufficiently large n. Putting

$$h = \frac{1}{2(B-A)} \exp(-\varrho(B-A)) \cdot I_{[A,B]}\,,$$

where $I_{[A,B]}$ is the indicator function of the interval $[A, B]$, we obtain

$$\lim \|(P_T^n f - h)^-\|_1 = 0 \quad \text{whenever } f \in H.$$

The condition (3) is a simple consequence of linearity of P_T and the fact that $f = f^+ - f^-$ ($z^+ = \max(0, z)$). This completes the proof.

An other proof of Lemma 1 is given in [6].

Denote by W the space of all integrable functions with bounded variation over R and such that

$$\int_{-\infty}^{+\infty} f\,dm = 0.$$

In W we introduce the norm: $\|f\|_W = \|f\|_1 + \bigvee_{-\infty}^{+\infty} f.$

LEMMA 2. *Let T be as in Theorem. Then*

$$\|P_T^n f\|_1 \to 0 \quad \text{if} \quad n \to \infty$$

uniformly in the unit ball in W.

Proof. Assume it is not. Then there exists $\varepsilon > 0$ such that for any $k \in N$ there are $n(k) > k$ and f_k from the unit ball in W which satisfy $\|P_T^{n(k)} f_k\|_1 > \varepsilon$. By virtue of (5) there exists a constant $K_2 > 0$ such that $|P_T f_k(x)| \leqslant K_2 \omega(x)$ for every k, where $\omega(x)$ is from (ix). This implies that the sequence $\{P_T f_k\}$ is weakly compact in $L_1(m)$. Then we may choose a subsequence $\{P_T f_{k_j}\}$ of $\{P_T f_k\}$ such that $P_T f_{k_j} \to f_0 \in L_1(m)$ weakly, if $j \to \infty$. Now, by (4) there exists a constant K_3 such that $\bigvee_{-\infty}^{+\infty} P_T f_{k_j} \leqslant K_3$ for any f_{k_j}. Thus, there exists a subsequence $\{P_T f_{k_{j_i}}\}$ of $\{P_T f_{k_j}\}$ which is convergent in $\|\cdot\|_1$ to f_0.

Denoting by $k(i) = k_{j_i}$ and $l(i) = n(k(i))$, we have

$$\begin{aligned}\|P_T^{l(i)-1} f_0\|_1 &= \|P_T^{l(i)-1} f_0 - P_T^{l(i)-1} P_T f_{k(i)} + P_T^{l(i)-1} P_T f_{k(i)}\|_1 \\ &\geqslant \|P_T^{l(i)-1} P_T f_{k(i)}\|_1 - \|P_T^{l(i)-1} f_0 - P_T^{l(i)-1} P_T f_{k(i)}\|_1 \geqslant \varepsilon - \|P_T^{l(i)-1} f_0 - P_T^{l(i)-1} P_T f_{k(i)}\|_1.\end{aligned}$$

Since

$$\|P_T^{l(i)-1} f_0 - P_T^{l(i)-1} P_T f_{k(i)}\|_1 \leqslant \|P_T^{l(i)-1}\|_1 \|f_0 - P_T f_{k(i)}\|_1 \to 0$$

if $i \to \infty$, we obtain the contradiction with the conclusion of Lemma 1. The proof is ended.

LEMMA 3. *Let T be as in Theorem. Then there are constants $K > 0$ and $0 < s < 1$ such that*

$$\|P_T^n\|_W \leqslant s^n K \quad \text{for all } n,$$

where $\|P_T^n\|_W = \sup\{\|P_T^n f\|_1 + \bigvee_{-\infty}^{\infty} P_T^n f \colon \|f\| W = 1\}.$

Proof. Since for any $\bar{n}$

$$\|P_T^n\|_W \leqslant \|P_T^{2\bar{n}}\|_W^{\beta} \cdot K_4,$$

where $\beta = \frac{n}{2n}$ and $K_4 = \sup\{\|P_T^k\|_W : k = 0, 1, \ldots, 2n-1\}$, for the proof of the lemma it is sufficient to show that for $\alpha \in (0, 1)$ there exists $\bar{n} \in N$ such that

$$\|P_T^{2\bar{n}}\|_W \leqslant \alpha .$$

From (4) there exists a constant K_5 such that for all n

$$\bigvee_{-\infty}^{+\infty} P_T^n f \leqslant K_5 \tag{8}$$

whenever $f \in W$ and $\|f\|_W = 1$. By Lemma 1 we can choose $\bar{n}$ so large that $\|P_T^{\bar{n}} f\|_1 \leqslant \frac{\alpha}{3}$ and $\|P_T^{\bar{n}} f\|_1 \leqslant \frac{q-1}{qM}\frac{\alpha}{3}$ and $q^{-\bar{n}} K_5 \leqslant \frac{\alpha}{3}$ for any functions from the unit ball in W. Thus, by (8) and from the fact that (see (4))

$$\bigvee_{-\infty}^{+\infty} P_T^{2n} f \leqslant q^{-n} \bigvee_{-\infty}^{+\infty} P_T^n f + \frac{qM}{q-1} \|P_T^n f\|_1$$

we obtain $\|P_T^{2\bar{n}}\|_W \leqslant \frac{\alpha}{3} + \frac{\alpha}{3} + \frac{\alpha}{3} = \alpha$.

Setting $K = \alpha^{-1} K_4$ and $s = \sqrt[2\bar{n}]{\alpha}$ we have the conclusion of the lemma.

Proof of Theorem. Let $f \geqslant 0$ be an integrable function with bounded variation over R. By Lemma 3 we have

$$|P_T^n f - g_\mu \|f\|_1| = \|f\|_1 |P_T^n h| \leqslant \|f\|_1 s^n K \|h\|_W ,$$

where $h = \frac{f}{\|f\|_1} - g_\mu$. Therefore

$$|P_T^n f - g_\mu \|f\|_1| \leqslant \|f\|_1 s^n K\Big(\bigvee_{-\infty}^{\infty} h + \|h\|_1\Big) \leqslant \|f\|_1 s^n K \left(\frac{1}{\|f\|_1} \bigvee_{-\infty}^{\infty} f + \bigvee_{-\infty}^{+\infty} g_\mu + 2 \right)$$

$$\leqslant s^n K\Big(\bigvee_{-\infty}^{\infty} f + \|f\|_1 (K_1 + 2)\Big) \leqslant s^n K\Big(\bigvee_{+\infty}^{\infty} f + C\|f\|_1\Big),$$

where $C = K_1 + 2$. Thus, we have proved (1). The proof of (2) is similar.

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