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The Properties of the Runge-Kutta Method

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§ 1. This paper deals with the method of Runge-Kutta of the infinite order.

In the preceding paper [1] we have used the difference inequalities and we have obtained the error estimate for the Runge-Kutta method of the finite order:

(1.1) $$y_{n+1}-y_n = \sum_{j=1}^{m} \omega_j \cdot k_{jy}^{(n)},\ y(x_0) = y_0,$$

where $y_n = y(x_n)$ denotes the approximate solution at the nodal point $x = x_n$ and

(1.2) $$\begin{cases} k_{1y}^{(n)} = h\cdot f(x_n, y_n), \\ k_{iy}^{(n)} = h\cdot f(x_n+\alpha_i\cdot h,\ y_n+\sum_{s=1}^{i-1} \beta_{is}\cdot k_{sy}^{(n)}), \\ i = 2, 3, \ldots, m), \end{cases}$$

and the order of the method is $m = 4$.

Let us denote by $\varphi(x)$ the solution of the equation

(1.3) $$y' = f(x, y),$$

of the class C^5 passing through the point (x_0, y_0), and by $r_n = y_n - \varphi_n$ the error, where $\varphi_n = \varphi(x_n)$. Then, as in the paper [1], we have the error estimate

(1.4) $$|r_n| \leqslant \frac{\mu(h)}{\Lambda}\cdot(e^{nh\Lambda}-1)\,(n = 0, 1, \ldots, N),$$

at an appropriate interval $x_0 \leqslant x \leqslant \beta$, cf. [1] § 7, the constant Λ being defined by

(1.5) $$\Lambda = \mathcal{L}\cdot\left(1+\frac{1}{2!}\mathcal{L}h+\frac{1}{3!}\mathcal{L}^2h^2+\frac{1}{4!}\mathcal{L}^3h^3\right).$$

These formulas lead us to the Runge-Kutta method of the infinite order $m = \infty$ with the same mesh size h, where the solution $y_n = y(x_n)$, $y(x_0) = y_0$, of the difference equation is defined by the formula

(1,6) $$y_{n+1}-y_n = \sum_{j=1}^{\infty} \omega_j k_{jy}^{(n)},$$

and

$$(1.7)\qquad \begin{cases} k_{1y}^{(n)} = h\cdot f(x_n, y_n), \\ k_{iy}^{(n)} = h\cdot f(x_n+\alpha_i h, y_n+\sum_{s=1}^{i-1}\beta_{is}\cdot k_{jy}^{(n)}), \\ i = 2, 3, \ldots \end{cases}$$

Thus, the Runge-Kutta methods (1,1) (1,2) of the finite order $m<\infty$ could be obtained from the formula (1,6) (1,7) by the truncation of the series, the corresponding series being replaced by its partial sums.

We can expect that for example the formula (1,4) for the error estimate takes the form

$$(1.8)\qquad |r_n|\leqslant\frac{\mu(h)}{\Lambda}\cdot(e^{nh\Lambda}-1)(n=0, 1, \ldots, N),$$

with the constant Λ given by the infinite series:

$$(1.9)\qquad \Lambda=\mathscr{L}\cdot\left(1+\sum_{s=1}^{\infty}\frac{1}{s!}\cdot\mathscr{L}^{s-1}h^{s-1}\right).$$

Many other questions arise in connection with the Runge-Kutta method (1,6) (1,7) of the infinite order.

It is significant how little assumptions should be made upon the right-hand member of the differential equation (1,3) in order to obtain the location of the solution y_n of the difference equation (1,6) (1,7), cf. Theorem 1.

Obviously if other properties of the method (1,6) (1,7) should be obtained, the assumptions must be taken in an appropriate manner.

§ **2.** We shall impose classical restrictions on the right-hand member of the differential equation

$$(2.1)\qquad y'=f(x, y).$$

We shall assume that

1° $f(x, y)$ is continuous in the set Q:

$$(2.2)\qquad Q:\ |x-\xi|\leqslant k,\ |y-\eta|\leqslant k,$$

2° $f(x, y)$ satisfies the Lipschitz condition

$$(2.3)\qquad |f(x,y)-f(x,\bar{y})|\leqslant\mathscr{L}\cdot|y-\bar{y}|,$$

$$\text{for}\quad (x,y)\in Q,\ (x,\bar{y})\in Q,\ 0<\mathscr{L}=\text{const}.$$

In addition we shall assume that

$$(2.4)\qquad |f(x,y)|\leqslant M,\quad \text{for } (x,y)\in Q,\ 0<M=\text{const},$$

and we denote

$$h_1 = \frac{k}{2M+1}. \tag{2.5}$$

Then, for an arbitrary point (x_0, y_0) in the set q:

$$q\colon\ |x-\xi|\leqslant h_1,\ |y-\eta|\leqslant h_1, \tag{2.6}$$

there exists the unique solution $y = \varphi(x)$ passing through the point $(x_0, y_0) \in q$, and defined in the interval $|x-\xi|\leqslant h_1$.

§ **3.** In regard to the class of admissible coefficients ω_j, α_i, β_{is}, in the difference equation (1,6) (1,7) we shall assume in this paper that

1° the numbers α_i satisfy the condition $0<\alpha_i\leqslant 1$ $(i = 2, 3, \ldots)$ and

$$\sum_{j=1}^{\infty} \omega_j = 1, \quad \sum_{j=1}^{\infty} |\omega_j| < +\infty. \tag{3.1}$$

2° If we introduce the numbers

$$\begin{cases} \gamma_0 = M\cdot \sum_{j=1}^{\infty} |\omega_j|, \\ \gamma_l = M\cdot \sum_{s=1}^{\infty} |\beta_{l+1,s}|, \\ (l = 1, 2, 3, \ldots), \end{cases} \tag{3.2}$$

then there exists the finite number γ:

$$\gamma = \sup_l \gamma_l. \tag{3.3}$$

§ **4.** We have the inequality $\gamma\geqslant M$, since $\sum_{j=1}^{\infty} \omega_j = 1$ and $1\leqslant \sum_{j=1}^{\infty} |\omega_j|$, which yields $M\leqslant M\sum_{j=1}^{\infty} |\omega_j|\leqslant \gamma_0\leqslant \gamma$.

Let us denote by (ξ_1, η_1) the point of intersection of the straight line $y = y_0 + \gamma\cdot(x-x_0)(x\geqslant x_0)$ with the boundary ∂P of the rectangle P:

$$P\colon\ |x-\xi|\leqslant h_1,\ |y-\eta|\leqslant k, \tag{4.1}$$

and by (ξ_2, η_2) the point of intersection of the line $y = y_0-\gamma\cdot(x-x_0)(x\geqslant x_0)$ with the boundary ∂P.

Let us denote

$$\beta = \min(\xi+h_1, \xi_1, \xi_2), \tag{4.2}$$

$$T\colon\ x_0\leqslant x\leqslant \beta,\ |y-y_0|\leqslant \gamma\cdot(x-x_0). \tag{4.3}$$

Thus the triangle T is contained in the rectangle P.

Let us introduce the nodal points

$$\begin{cases} x_0 < x_1 < x_2 < \dots < x_N = \beta, \ \beta - x_0 = N \cdot h, \\ x_{n+1} - x_n = h \ (n = 0, 1, \dots, N-1), \end{cases} \tag{4.4}$$

in the interval $x_0 \leqslant x \leqslant \beta$. We assume that the mesh size h is constant. It might be changed from one nodal point to another without changing the main idea of the paper.

§ 5. In this section we shall prove the theorem on the location of the solution y_n of the difference equation (1.6) (1.7).

THEOREM 1. *Under the assumptions of* § 2 *and* § 3 *and notations of* § 4

1° *the solution* y_n *of the difference equation* (1.6) (1.7) *satisfying the initial condition* $y(x_0) = y_0$ *is defined for* $n = 0, 1, 2, \dots, N$;

2° *the points* (x_n, y_n) $(n = 0, 1, \dots, N)$ *are in the triangle* T, *cf.* (4.3);

3° *the right-hand members of the formula* (1.7) *are defined for* $n = 0, 1, \dots, N$.

Proof. We shall proceed by induction.

a) The point (x_0, y_0) belongs to the triangle T.

b) Let us suppose that for the fixed natural number p $(0 \leqslant p \leqslant N-1)$ the value y_p is defined and

$$(x_p, y_p) \in T. \tag{5.1}$$

We shall prove that the next value y_{p+1} is defined, and

$$(x_{p+1}, y_{p+1}) \in T. \tag{5.2}$$

For this purpose we shall verify first that

$$\begin{cases} (x_p + \alpha_i h, \ y_p + \sum_{s=1}^{i-1} \beta_{is} \cdot k_{sy}^{(p)}) \in T, \\ (i = 2, 3, 4, \dots). \end{cases} \tag{5.3}$$

In fact, the relation (5.3) holds for $i = 2$, since we have

$$|\beta_{21} \cdot k_{1y}^{(p)}| \leqslant |\beta_{21}| \cdot h \cdot |f(x_p, y_p)| \leqslant |\beta_{21}| \cdot hM = h \cdot \gamma_1 \leqslant h \cdot \gamma, \tag{5.4}$$

because of the induction assumption (5.1) and the definition of the numbers γ_1 and γ, cf. (3.2) (3.3).

The inequality (5.4) means that the relation (5.3) holds for $i = 2$, i.e.

$$(x_p + \alpha_2 \cdot h, \ y_p + \beta_{21} \cdot k_{1y}^{(p)}) \in T, \tag{5.5}$$

hence the value $f(x, y)$ at the point (5.5) is defined as well as the value $k_{2y}^{(p)}$.

In a similar way we verify that (5.3) holds for $i = 3$:

(5.6) $$(x_p+\alpha_3 h, y_p+\sum_{s=1}^{2}\beta_{3s}\cdot k_{sy}^{(p)}) \in T.$$

In fact, we have

(5.7) $$|\sum_{s=1}^{2}\beta_{3s}\cdot k_{sy}^{(p)}| \leqslant \sum_{s=1}^{2}|\beta_{3s}|\cdot hM = h\gamma_2 \leqslant h\gamma .$$

The inequality (5.7) means that the relation (5.6) holds, hence the value $f(x, y)$ at the point (5.6) and the value $k_{3y}^{(p)}$ are defined.

With the aid of induction we verify in a similar way that the points (5.3) are in the triangle T for all values $i = 2, 3, 4, \ldots$, hence the values $f(x, y)$ at the points (5.3) are defined as well as the values $k_{2y}^{(p)}, k_{3y}^{(p)}, \ldots$

If we write now the difference equation (1.6) for $n = p$ we see that the value y_{p+1} is defined, because of the inequality $|k_{sy}^{(p)}| \leqslant h\cdot M$ $(s = 1, 2, \ldots)$ and the assumption (3.1).

In addition we have

(5.8) $$|y_{p+1}-y_p| = |\sum_{j=1}^{\infty}\omega_j\cdot k_{jy}^{(p)}| \leqslant \sum_{j=1}^{\infty}|\omega_j|\cdot hM \leqslant h\cdot\gamma_0 \leqslant h\cdot\gamma ,$$

which means that the relation (5.2) holds.

Thus the solution y_n of the difference equation (1.6) (1.7) satisfying the initial condition $y(x_0) = y_0$ is defined for $n = 0, 1, 2, \ldots, N$ and the points $(x_n, y_n)(n = 0, 1, \ldots, N)$ are in the triangle T.

This ends the proof of Theorem 1.

References

[1] Z. Kowalski, *Difference inequalities and error estimates for Runge-Kutta method,* Annales Polonici Mathematici, vol. XLII (1983), 149–158.

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