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A Difference Method for a Non-linear Elliptic Equation with Mixed Derivatives

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§ 1. The purpose of the following considerations is the difference method of solving the non-linear differential equation of elliptic type, cf. § 4, containing all derivatives of the second order of the function u:

$$f\left(x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_n^2}\right) = 0 . \tag{1.1}$$

The proof of convergence of that method as given in the Theorem 2, cf. § 9, is based on the results of the preceding paper [1].

In order to obtain the error estimate which is more adequate for computing the solution of the boundary problem, another proof of convergence has been given as well as two error estimates, cf. Theorem 5, § 18.

The second error estimate as given in Theorem 5 can be obtained with the aid of the precise location of errors r^M, cf. Lemma 1, § 12.

The question of existence of the solution v^M of the difference equation, cf. (4.12), as well as the properties (4.10) for difference quotients, are by all means the hardest to settle, cf. [2], and we therefore postpone discussing it until latter.

§ 2. Let us denote by Q the set of points $x \in R^n$, $x = (x_1, \ldots, x_n)$:

$$Q\colon\ 0 \leqslant x_j \leqslant \sigma \ (j = 1, \ldots, n),\ 0 < \sigma = \text{const} . \tag{2.1}$$

We shall denote by M the sequence of indices

$$M = (m_1, m_2, \ldots, m_n),\ 0 \leqslant m_j \leqslant N \ (j = 1, 2, \ldots, n) , \tag{2.2}$$

and by x^M the nodal point with coordinates

$$x^M = (x_1^M, x_2^M, \ldots, x_n^M) , \tag{2.3}$$

where $x_j^M = m_j h$ $(j = 1, \ldots, n)$, $0 < h = \sigma/N$, N being the natural number.

We shall introduce also the nodal points in the set Q characterized by the following sequences of indices:

$$\begin{aligned}
&j(M) = (m'_1, \ldots, m'_n),\ m'_j = m_j+1,\ m'_i = m_i \quad \text{for } i \neq j, \\
(2.4)\qquad &-j(M) = (m'_1, \ldots, m'_n),\ m'_j = m_j-1,\ m'_i = m_i \quad \text{for } i \neq j, \\
&(i = 1, \ldots, n;\ j = 1, \ldots, n),
\end{aligned}$$

and for $i \neq j$:

$$\begin{aligned}
&ij(M) = (m'_1, \ldots, m'_n),\ m'_i = m_i+1,\ m'_j = m_j+1, \\
&-ij(M) = (m'_1, \ldots, m'_n),\ m'_i = m_i-1,\ m'_j = m_j+1, \\
(2.5)\qquad &-i-j(M) = (m'_1, \ldots, m'_n),\ m'_i = m_i-1,\ m'_j = m_j-1, \\
&i-j(M) = (m'_1, \ldots, m'_n),\ m'_i = m_i+1,\ m'_j = m_j-1,
\end{aligned}$$

where $m'_s = m_s$ in the formula (2.5) for $s = 1, \ldots, n$; $s \neq i$, $s \neq j$, cf. Fig. 1.

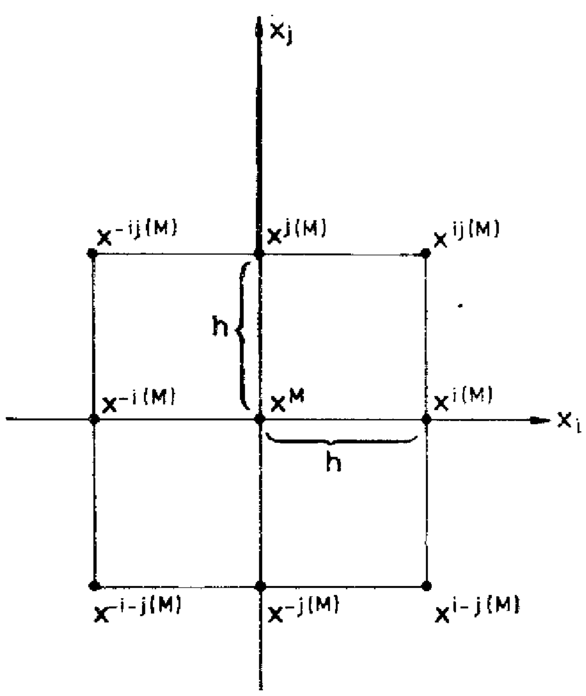

Fig. 1. The nodal points x^M, $x^{i(M)}$, $x^{ij(M)}$, ... For the sake of simplicity the nodal point x^M has been located at the origin.

The nodal point $x^{ij(M)}$ can be denoted also by $x^{ji(M)}$ since we define

$$\begin{aligned}
(2.6)\qquad &ij(M) = ji(M),\ -ij(M) = j-i(M),\ -i-j(M) = -j-i(M), \\
&i-j(M) = -ji(M), \quad \text{for } i \neq j\ (i, j = 1, \ldots, n).
\end{aligned}$$

We denote by $\operatorname{int} Q$ the set of nodal points (2.3) which belong to the interior of the set Q, cf. (2.1), and by $\operatorname{sym} A$ the set of nodal points x^M such that $x^M \in \operatorname{int} Q$ and $x^{M^*} \in \operatorname{int} Q$ simultaneously, x^{M^*} and x^M being symmetric with respect to the nodal point x^A.

§ **3.** Let us denote by u^M the value of the function $u(x)$ at the nodal point x^M. We shall consider the difference quotients

(3.1) $$u_+^{Mj} = \frac{1}{h}\cdot(u^{j(M)} - u^M),\ u_-^{Mj} = \frac{1}{h}\cdot(u^M - u^{-j(M)}),$$

(3.2) $$u^{Mj} = \frac{1}{2h}\cdot(u^{j(M)} - u^{-j(M)}),$$

for the first partial derivatives, and the difference quotients

(3.3) $$\begin{aligned} u^{Mjj} &= h^{-2}\cdot(u^{j(M)} - 2u^M + u^{-j(M)}), \\ u^{Mij} &= \tfrac{1}{4}\cdot h^{-2}(u^{ij(M)} - u^{-ij(M)} - u^{i-j(M)} + u^{-i-j(M)})\ (i \neq j) \end{aligned}$$

for the second derivatives.

From the definitions (3.1) (3.2) (3.3) we obtain

(3.4) $$u^{Mj} = \frac{1}{2}\cdot(u_+^{Mj} + u_-^{Mj}),\ u^{Mjj} = \frac{1}{h}\cdot(u_+^{Mj} - u_-^{Mj}).$$

We shall also use the difference quotients u_{++}^{Mij}, u_{-+}^{Mij}, u_{--}^{Mij}, u_{+-}^{Mij}, cf. Fig. 1:

(3.5) $$\begin{aligned} u_{++}^{Mij} &= h^{-2}(u^{ij(M)} - u^{j(M)} - u^{i(M)} + u^M), \\ u_{-+}^{Mij} &= h^{-2}(u^{j(M)} - u^{-ij(M)} - u^M + u^{-i(M)}), \\ u_{--}^{Mij} &= h^{-2}(u^M - u^{-i(M)} - u^{-j(M)} + u^{-i-j(M)}), \\ u_{+-}^{Mij} &= h^{-2}(u^{i(M)} - u^M - u^{i-j(M)} + u^{-j(M)}), \quad \text{for } i \neq j. \end{aligned}$$

From (3.5) and the definition of u^{Mij}, cf. (3.3), it follows that

(3.6) $$u^{Mij} = \tfrac{1}{4}\cdot(u_{++}^{Mij} + u_{-+}^{Mij} + u_{--}^{Mij} + u_{+-}^{Mij}).$$

We introduce also the vector $u^{M\Delta}$ with coordinates

(3.7) $$u^{M\Delta} = (u^{M1}, u^{M2}, \dots, u^{Mn}),$$

and the $n \times n$ matrix $u^{M\Box}$

(3.8) $$u^{M\Box} = (u^{Mij}).$$

§ **4.** Throughout the rest of this paper we shall use the following Assumptions H:

Assumptions *H*. 1) Let us assume that the scalar function

$$f(x, u, q, w),\ x = (x_1, \dots, x_n) \in R^n,\ u \in R^1,\ q = (q_1, \dots, q_n) \in R^n,\ w = (w_{ij}),$$

is of the class C^1 in the set (4.1):

(4.1) $$\begin{aligned} &0 \leqslant x_j \leqslant \sigma,\ -\infty < u < +\infty,\ -\infty < q_j < +\infty, \\ &-\infty < w_{ij} < +\infty\ (i, j = 1, \dots, n),\ 0 < \sigma = \text{const}, \end{aligned}$$

and satisfies the condition

$$f_{w_{ij}} = f_{w_{ji}}, \tag{4.2}$$

in the set (4.1), where $f_{w_{ij}} = \partial f/\partial w_{ij}$ $(i, j = 1, \ldots, n)$.

2) Let us denote by u_x the vector $u_x = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$ and by u_{xx} the matrix $u_{xx} = (u_{x_i x_j})$. We assume that the nonlinear differential equation

$$f(x, u, u_x, u_{xx}) = 0, \tag{4.3}$$

is of elliptic type which means that 1° the quadratic form (4.4) is positive at every point of the set (4.1):

$$\sum_{i,j=1}^{n} f_{w_{ij}} \lambda_i \lambda_j > 0 \quad \text{for } \lambda_i \neq 0,\ \lambda_j \neq 0\ (i, j = 1, \ldots, n), \tag{4.4}$$

and 2° the derivatives $f_{w_{ij}}, f_{q_j}, f_u$ satisfy the conditions

$$|f_{w_{ij}}| \leqslant \gamma,\ |f_{q_j}| \leqslant \beta,\ f_u \leqslant \eta < 0 \text{ in the set (4.1)} \tag{4.5}$$

for $i, j = 1, \ldots, n$, γ, β and η being constant.

3) We assume that the function $u = u(x) (x \in Q)$ satisfies the following conditions 1° $u(x)$ is of the class C^2 for $x \in Q$, cf. (2.1);

2° the derivatives $u_{x_i x_j}$ satisfy the Lipschitz condition

$$|u_{x_i x_j}(x) - u_{x_i x_j}(x')| \leqslant \tfrac{1}{2} \cdot L |x_s - x_s'| \ (i, j = 1, \ldots, n;\ x \in Q,\ x' \in Q), \tag{4.6}$$

the points x and x' being on the x_s-axis $(s = 1, \ldots, n)$,

$$x = (x_1, \ldots, x_n),\ x' = (x_1', \ldots, x_n'),\ x_s \neq x_s',\ x_p = x_s'\ (p \neq s;\ p = 1, \ldots, n).$$

3° the inequalities

$$|u_{x_i x_j}| \leqslant \tfrac{1}{2} A \quad (i, j = 1, \ldots, n), \tag{4.7}$$

hold for $x \in Q$, the constant A being independent of x.

4° the function $u(x) (x \in Q)$ is the solution of the non-linear differential equation of elliptic type

$$f(x, u, u_x, u_{xx}) = 0, \tag{4.8}$$

and takes prescribed values $\varphi(x)$ at the boundary ∂Q of the set Q:

$$u(x) = \varphi(x), \quad \text{for } x \in \partial Q, \tag{4.9}$$

$\varphi(x)$ being continuous for $x \in \partial Q$.

4) We assume that the discrete function v^M satisfies the following conditions 1° v^M is defined at the nodal points $x^M \in Q$,

2° the difference quotients v^{Mij} satisfy the inequalities

$$\begin{aligned} &|v^{Mjj} - v^{Pjj}| \leqslant \tfrac{1}{2} \cdot hL,\ |v^{Mij}_{++} - v^{Pij}_{++}| \leqslant \tfrac{1}{2} \cdot hL, \\ &|v^{Mij}_{-+} - v^{Pij}_{-+}| \leqslant \tfrac{1}{2} \cdot hL,\ |v^{Mij}_{--} - v^{Pij}_{--}| \leqslant \tfrac{1}{2} \cdot hL, \\ &|v^{Mij}_{+-} - v^{Pij}_{+-}| \leqslant \tfrac{1}{2} \cdot hL, \quad \text{for } h > 0, \end{aligned} \tag{4.10}$$

at the nodal points x^M and x^P, $P = s(M)(s = \pm 1, \pm 2, ..., \pm n)$, the distance between x^M and x^P being h in the direction of the x_s-axis.

3° the inequalities

$$|v^{Mij}| \leqslant \tfrac{1}{2} A \ (i, j = 1, ..., n), \tag{4.11}$$

hold for $x^M \in \operatorname{int} Q$.

4° the function $v^M (x^M \in Q)$ is the solution of the difference equation

$$f(x^M, u^M, u^{M\Delta}, u^{M\Box}) = 0, \quad \text{for } x^M \in \operatorname{int} Q, \tag{4.12}$$

and takes on prescribed values $\varphi(x^M)$ at the boundary ∂Q of the set Q:

$$v^M = \varphi(x^M), \quad \text{for } x^M \in \partial Q. \tag{4.13}$$

5) The characteristic roots s_k, $s_k > 0$ $(k = 1, ..., n)$ of the form $\sum_{i,j=1}^{n} f_{w_{ij}} \lambda_i \lambda_j$ are bounded:

$$0 < \delta_1 \leqslant s_k \leqslant \delta_2 \ (k = 1, ..., n) \tag{4.14}$$

the constants δ_1 and δ_2 being independent of the point of the set (4.1).

§ 5. Remark 1. Let us denote

$$r^M = u^M - v^M, \quad \text{for } x^M \in Q. \tag{5.1}$$

From Assumptions (4.6) and (4.10) it follows that the errors r^M satisfy the inequalities

$$\begin{gathered} |r^{Mij} - r^{Pij}| \leqslant hL, \ |r^{Mij}_{++} - r^{Pij}_{++}| \leqslant hL, \\ |r^{Mij}_{-+} - r^{Pij}_{-+}| \leqslant hL, \ |r^{Mij}_{--} - r^{Pij}_{--}| \leqslant hL, \\ |r^{Mij}_{+-} - r^{Pij}_{+-}| \leqslant hL, \ (i \neq j), \quad \text{for } h > 0, \end{gathered} \tag{5.2}$$

at the nodal points x^M and x^P, $P = s(M)$ $(s = \pm 1, \pm 2, ..., \pm n)$, the distance between x^M and x^P being h in the direction of the x_s-axis.

From (4.7) and (4.11) we obtain also

$$|r^{Mij}| \leqslant A, \quad \text{for } x^M \in \operatorname{int} Q \ (i, j = 1, ..., n). \tag{5.3}$$

§ 6. Remark 2. The solution $u(x)$ of the equation (4.8) satisfies the following equation at the point x^M

$$f(x^M, u^M, u^{M\Delta}, u^{M\Box}) = \varepsilon^M, \quad \text{for } x^M \in \operatorname{int} Q, \tag{6.1}$$

ε^M being dependent of x^M.

Let us denote

$$\varepsilon(h) = \max_{x^M \in \operatorname{int} Q} |\varepsilon^M|. \tag{6.2}$$

It can be seen that

(6.3) $$\varepsilon(h) \to 0, \quad \text{as} \quad h \to 0,$$

because f is of the class C^1 and $u(x)$ is of the class C^2, cf. Assumptions H.

§ 7. We shall make use of the following theorems, cf. [1], proved under the Assumptions K:

ASSUMPTIONS K. 1) Let us suppose that the function r^M is defined at the nodal points x^M of the set Q, cf. (2.1).

2) There exists the positive constant $L>0$, independent of the mesh size h such that the difference quotients r^{Mij} $(i \neq j)$ and r^{Mjj} satisfy the conditions (5.2) and (5.3), cf. § 5.

3) Let us consider the difference inequality for the function r^M:

(7.1) $$\sum_{i,j=1}^{n} a_{ij}^M \cdot r^{Mij} + \sum_{j=1}^{n} b_j^M \cdot r^{Mj} + c^M \cdot r^M \geqslant -\varepsilon,$$

for $x^M \in \operatorname{int} Q$, $0<\varepsilon = \text{const}$. We suppose that 1° the inequality (7.1) is of elliptic type which means that $\sum_{i,j=1}^{n} a_{ij}^M \lambda_i \lambda_j$ $(x^M \in Q)$ is the positive definite quadratic form, and

2° the coefficients a_{ij}^M, b_j^M, c^M are bounded:

(7.2) $$|a_{ij}^M| \leqslant \gamma, \quad |b_j^M| \leqslant \beta, \quad c^M \leqslant \eta < 0,$$

the constants γ, β and η being independent of the mesh size h.

4) We suppose that $r^M = 0$ for $x^M \in \partial Q$, where ∂Q denotes the boundary of the set Q.

5) The characteristic roots $s_k^M, s_k^M>0$ $(k = 1, \ldots, n)$, of the form $\sum_{i,j=1}^{n} a_{ij}^M \lambda_i \lambda_j$ are bounded:

(7.3) $$0<\delta_1 \leqslant s_k^M \leqslant \delta_2 \quad (k = 1, \ldots, n),$$

the constants δ_1 and δ_2 being independent of the mesh size h.

THEOREM A. *Let us suppose that the function r^M satisfies the Assumptions K and the quantity ε in* (7.1) *depends on h*:

(7.4) $$0<\varepsilon = \varepsilon(h) \to 0, \quad \textit{as} \quad h \to 0.$$

Let us denote

(7.5) $$r^A = \max_{x^M \in Q} r^M, \; A = A(h).$$

Under these assumptions

(7.6) $$r^{A(h)} \to 0, \quad \textit{as} \quad h \to 0.$$

THEOREM B. *Let us suppose that the function r^M satisfies Assumptions K, the inequality (7.1) being replaced by the inequality*

(7.7) $$\sum_{i,j=1}^{n} a_{ij}^M \cdot r^{Mij} + \sum_{j=1}^{n} b_j^M \cdot r^{Mj} + c^M \cdot r^M \leqslant +\varepsilon,$$

where $0<\varepsilon = \varepsilon(h) \to 0$, *as* $h \to 0$.

Let us denote

(7.8) $$r^B = \min_{x^M \in Q} r^M, \quad B = B(h).$$

Under these assumptions we have

(7.9) $$r^{B(h)} \to 0, \quad as \quad h \to 0.$$

§ 8. We shall prove now Theorem 1.

THEOREM 1. *Let us suppose that the Assumptions H are fulfilled. Suppose in addition that the error r^M is defined by the formula* (5.1) *and the quantity* $\varepsilon(h)$ *by* (6.1) *and* (6.2).

Under these assumptions r^M satisfies the difference inequalities of elliptic type:

(8.1) $$\sum_{i,j=1}^{n} a_{ij}^M \cdot r^{Mij} + \sum_{j=1}^{n} b_j^M \cdot r^{Mj} + c^M \cdot r^M \geqslant -\varepsilon(h),$$

(8.2) $$\sum_{i,j=1}^{n} a_{ij}^M \cdot r^{Mij} + \sum_{j=1}^{n} b_j^M \cdot r^{Mj} + c^M \cdot r^M \leqslant +\varepsilon(h),$$

for $x^M \in \operatorname{int} Q$. *In the formula* (8.1) *and* (8.2), $\varepsilon(h)$ *satisfies the relation* (6.3), *and the coefficients* a_{ij}^M, b_j^M, c^M *are defined by*

(8.3) $$a_{ij}^M = f_{w_{ij}}(\sim), \; b_j^M = f_{q_j}(\sim), \; c^M = f_u(\sim),$$

the derivatives being taken at a suitable point $(\sim)$.

Proof. From (6.1) and (4.12) we obtain

(8.4) $$f(x^M, u^M, u^{M\Delta}, u^{M\Box}) - f(x^M, v^M, v^{M\Delta}, v^{M\Box}) = \varepsilon^M,$$

for $x^M \in \operatorname{int} Q$. Now we can apply the mean value theorem to the left-hand member of (8.4) and we get by (3.7) and (3.8):

(8.5) $$f_u(\sim) \cdot r^M + \sum_{j=1}^{n} f_{q_j}(\sim) \cdot r^{Mj} + \sum_{i,j=1}^{n} f_{w_{ij}}(\sim) \cdot r^{Mij} = \varepsilon^M,$$

the derivatives being taken at the suitable point $(\sim)$.

(8.5) can be written in the form

(8.6) $$\sum_{i,j=1}^{n} a_{ij}^M \cdot r^{Mij} + \sum_{j=1}^{n} b_j^M \cdot r^M + c^M \cdot r^M = \varepsilon^M,$$

because of (8,3), hence from the equality (8.6) and the definition of $\varepsilon(h)$, cf. (6.2), we obtain two inequalities:

$$\sum_{i,j=1}^{n} a_{ij}^{M}\cdot r^{Mij}+\sum_{j=1}^{n} b_{j}^{M}\cdot r^{Mj}+c^{M}\cdot r^{M}\geqslant-\varepsilon(h)\,, \tag{8.7}$$

and

$$\sum_{i,j=1}^{n} a_{ij}^{M}\cdot r^{Mij}+\sum_{j=1}^{n} b_{j}^{M}\cdot r^{Mj}+c^{M}\cdot r^{M}\leqslant+\varepsilon(h)\,, \tag{8.8}$$

for $x^M \in \operatorname{int} Q$.

This ends the proof of Theorem 1.

§ 9. THEOREM 2. *Let us suppose that the Assumptions H are fulfilled and u^M is the solution of the non-linear differential equation of elliptic type* (4.8), *and satisfies the boundary condition* (4.9).

Suppose in addition that v^M is the solution of the difference equation (4.12) *and satisfies the boundary condition* (4,13), *and let us denote by r^M the error $r^M = u^M - v^M$.*

Under these assumptions the difference method is convergent i.e.

$$r^M \to 0,\quad as \quad h \to 0 \ (x^M \in Q)\,. \tag{9.1}$$

Proof. It is sufficient to verify that r^M satisfies the assumptions of the Theorem A and Theorem B.

For this purpose we shall consider the Assumptions K successively, cf. § 7.

From Remark 1, cf. § 5, it follows that part 1) and 2) of Assumptions K are fulfilled.

From Theorem 1, cf. (8.1) (8.2), we obtain the inequalities (7.1) and (7.7).

From the definition (8.3) and the Assumptions H, cf. (4.4) (4.5), it follows that the conditions (7.2) for the coefficients a_{ij}^M, b_j^M, c^M are satisfied and $\sum_{i,j=1}^{n} a_{ij}^M \lambda_i \lambda_j$ is the positive defined quadratic form. Therefore the part 3) of Assumptions K is fulfilled.

Part 4) of Assumptions K is also fulfilled: $r^M = u^M - v^M = 0$ for $x^M \in \partial Q$, because of the relations (4.9) and (4.13).

Part 5) of Assumptions K holds, because of the definition (8.3) and part 5) of Assumptions H.

Hence we can apply Theorem A and Theorem B and we obtain the convergence:

$$\max_{x^M \in Q} r^M \to 0,\ \min_{x^M \in Q} r^M \to 0, \quad \text{as} \quad h \to 0\,, \tag{9.2}$$

and

$$r^M \to 0, \quad \text{as} \quad h \to 0 \ (x^M \in Q)\,. \tag{9.3}$$

This ends the proof of Theorem 2.

§ **10.** The error estimate can be obtained from Theorem 1, Theorem 2 and Theorem 3 of the paper [1], under additional assumption:

$$V \subset Q,\ V' \subset Q,\ |x^b - x^A| \geqslant h^\alpha,\ |x^{b'} - x^B| \geqslant h^\alpha, \tag{10.1}$$

for $x^b \in \partial V$, $x^{b'} \in \partial V'$, where the sets V and V' are defined by

$$V = \{x\colon |x_j - x_{j_M}^M| \leqslant h, h.\ |m-a| \leqslant h^\alpha\ (j = 1, \ldots, n)\}, \tag{10.2}$$

$$V' = \{x\colon |x_j - x_j^M| \leqslant h, h.\ |m-b| \leqslant h^\alpha\ (j = 1, \ldots, n)\}. \tag{10.3}$$

In the formula (10.2) (10.3) the nodal points x^A, $A = A(h)$, $A = (a_1, \ldots, a_n)$ and x^B, $B = B(h)$, $B = (b_1, \ldots, b_n)$, satisfy the conditions

$$r^A = \max_{x^M \in Q} r^M,\ r^B = \min_{x^M \in Q} r^M, \tag{10.4}$$

and α is the constant, $0<\alpha<1$.

It is the assumption (10.1) which provides the error estimate unsatisfactory from the numerical point of view. We could hardly expect that (10.1) can be verified conveniently when the boundary value problem (4.8) (4.9) is solved numerically.

Therefore we shall give the another error estimate, cf. Theorem 5 part 3°, which is more adequate for really computing the solution of the boundary problem.

In particular, we shall prove that under the suitable choice of the number α the sets V and V' can be defined in a different way so as to obtain (10.1) for every $h>0$.

It is worth noticing that the corresponding consideration furnishes the second proof of convergence of the difference method, which does not involve the proof by a contradiction as in Theorem 2 in the paper [1]. This provides a better insight into the mechanism of the difference method.

§ **11.** Assumptions K_1. We assume that the function r^M is defined for $x^M \in Q$ and satisfies the conditions

$$r^M = 0, \quad \text{for } x^M \in \partial Q, \tag{11.1}$$

$$\begin{cases} |r_+^{Mj}| \leqslant h.\vartheta, & \text{for } x^M \in \partial Q,\ x^{j(M)} \in \operatorname{int} Q \quad (j = 1, \ldots, n), \\ |r_-^{Mj}| \leqslant h.\vartheta, & \text{for } x^M \in \partial Q,\ x^{-j(M)} \in \operatorname{int} Q \quad (j = 1, \ldots, n), \end{cases} \tag{11.2}$$

and

$$|r^{Mij}| \leqslant A, \quad \text{for } x^M \in \operatorname{int} Q \quad (i, j = 1, \ldots, n), \tag{11.3}$$

where $0<\vartheta = \text{const}$.

§ **12.** We shall prove now Lemma 1 on the location of the function r^M for $x^M \in Q$.

Lemma 1. *Let us suppose that the function* r^M *satisfies the Assumptions* K_1 *and let us consider the function*

$$z = (\tfrac{1}{2} \cdot A + \vartheta) \cdot y^2,\ 0 \leqslant y < +\infty. \tag{12.1}$$

Under these assumptions

$$|r^M| \leqslant (\tfrac{1}{2}\cdot\Lambda + \vartheta)\cdot y^2, \tag{12.2}$$

where y denotes the Euclidean distance $\varrho(x^M, \partial Q)$ of the point $x^M \in Q$ from the boundary ∂Q, $y = \varrho(x^M, \partial Q)$. In the formula (12.2) *we have the strong inequality for $\varrho(x^M, \partial Q) > 0$, cf. Fig.* 2.

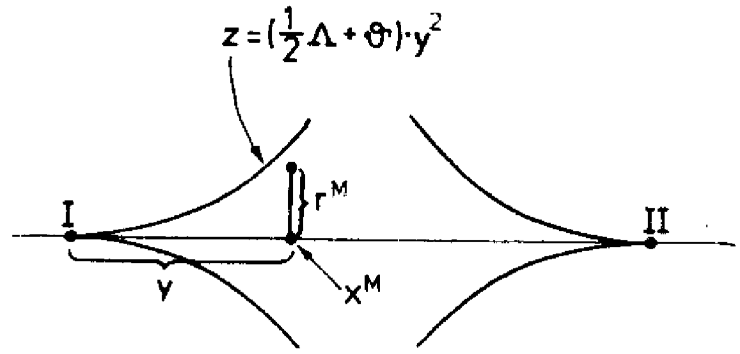

Fig. 2. The location of the function r^M. The set Q = I, II as seen from the edge in the case $n = 2$

The proof of Lemma 1 is simple. But we shall use it in subsequent papers, therefore let us see once only how this can be done.

Proof. Let us denote by x^{M_s} $(s = 1, 2, \dots)$ the finite sequence of nodal points that are lying on the straight line parallel to the x_j-axis $(j = 1, \dots, n)$.

Let us assume that $x^{M_1} \in \partial Q$, $x^{M_s} \in \operatorname{int} Q$ $(s = 2, 3, \dots)$, $0 \leqslant x_j^{M_s} \leqslant \frac{1}{2}\cdot\sigma$ $(j = 1, 2, \dots, n)$, the distance between two successive nodal points being h.

Then we have

$$r^{M_s} = \sum_{i=2}^{s} (r^{M_i} - r^{M_{i-1}}) \quad (s = 2, 3, \dots), \tag{12.3}$$

since $r^{M_1} = 0$ because of the assumption (11.1).

But from the definition (3.1) of difference quotients it follows that

$$r^{M_i} - r^{M_{i-1}} = h . r_+^{M_{i-1}j} \quad (i = 2, 3, \dots), \tag{12.4}$$

hence

$$r^{M_s} = h\cdot\sum_{i=2}^{s} r_+^{M_{i-1}j} \quad (i = 2, 3, \dots) \tag{12.5}$$

The formula (12.5) can be rewritten in the form

$$r^{M_s} = h\cdot\sum_{i=3}^{s} r_+^{M_{i-1}j} + h\cdot r_+^{M_1 j} \quad (s = 3, 4, \dots). \tag{12.6}$$

If we write the terms on the right-hand side in the form

$$r_+^{M_{i-1}j} = \sum_{k=2}^{i-1} (r_+^{M_k j} - r_+^{M_{k-1}j}) + r_+^{M_1 j} \quad (s = 3, 4, \dots), \tag{12.7}$$

and if we take into account the definition (3.4) of the difference quotients of the second order, then we find that

$$r_+^{M_k j} - r_+^{M_{k-1}j} = h\cdot r^{M_k jj} \quad (k = 2, 3, \dots), \tag{12.8}$$

hence from (12.8) and (12.7) we obtain

(12.9) $$r_+^{M_{i-1}j} = h\cdot \sum_{j=2}^{i-1} r^{M_k jj} + r_+^{M_1 j} \quad (i = 3, 4, \dots).$$

From (12.9) and (12.6) it follows that

(12.10) $$r^{M_s} = h.[\sum_{i=3}^{s} (h\cdot \sum_{k=2}^{i-1} r^{M_k jj} + r_+^{M_1 j})] + h\cdot r_+^{M_1 j} =$$
$$= h^2\cdot [\sum_{i=3}^{s} \sum_{k=2}^{i-1} r^{M_k jj}] + h\cdot (s-1)\cdot r_+^{M_1 j}, \quad (s = 3, 4, \dots).$$

But we have

(12.11) $$\sum_{i=3}^{s} \sum_{k=2}^{i-1} r^{M_k jj} = \sum_{l=2}^{s-1} (s-l)\cdot r^{M_l jj} \quad (s = 3, 4, \dots),$$

hence

(12.12) $$|r^{M_s}| \leqslant h^2\cdot A\cdot \sum_{l=2}^{s-1} (s-l) + h^2\cdot (s-1)\cdot \vartheta \quad (s = 3, 4, \dots),$$

because of (12.11) (12.10) and the Assumptions K_1.

The formula $\sum_{l=2}^{s-1} (s-l) = \frac{1}{2}\cdot (s-1)(s-2)$, $(s = 3, 4, \dots)$, and (12.12) yields

(12.13) $$|r^{M_s}| \leqslant \tfrac{1}{2}\cdot h^2 A (s-1)(s-2) + h^2\cdot (s-1)\cdot \vartheta \quad (s = 3, 4, \dots).$$

In the formula (12.13) we can write $(s-1)(s-2) < s^2$, $s-1 < s^2$, therefore we have

(12.14) $$|r^{M_s}| < (\tfrac{1}{2}\cdot A + \vartheta)\cdot s^2 h^2 \quad (s = 3, 4, \dots).$$

Let us consider the function

(12.15) $$z = (\tfrac{1}{2}\cdot A + \vartheta)\cdot y^2, \ 0 \leqslant y < +\infty .$$

It can be seen from (12.15) and (12.14) that for $s = 3, 4, \dots$ we have

(12.16) $$|r^{M_s}| \leqslant (\tfrac{1}{2}\cdot A + \vartheta)\cdot y^2, \quad \text{for } y = hs \ (s = 3, 4, \dots).$$

From the assumption $|r_+^{M_1 j}| \leqslant h\cdot \vartheta$, cf. (11.2), it follows immediately that (12.16) remains valid for $s = 2$, i.e. for r^{M_2}. In fact,

(12.17) $$|r_+^{M_1 j}| = h^{-1}\cdot |r^{M_2} - r^{M_1}| \leqslant h\cdot \vartheta .$$

But $r^{M_1} = 0$, therefore $|r^{M_2}| \leqslant h^2\cdot \vartheta$. The inequality $h^2\cdot \vartheta < (\frac{1}{2}\cdot A + \vartheta)\cdot h^2$ implies that

(12.18) $$|r^{M_2}| < (\tfrac{1}{2}\cdot A + \vartheta)\cdot (2h)^2,$$

cf. (12.16).

If the nodal points x^{M_s} follow in the direction opposite to the direction of the x_j-axis $(j = 1, 2, \dots, n)$ beginning with x^{M_1} on the boundary ∂Q, then the calculations can be performed in a similar way.

This ends the proof of Lemma 1.