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Almost-Algebraic Sets

by Konrad CZAJA

Abstract. The concepts of almost-algebraic and essentially transcendental sets are introduced and the following results are proved:

A. Let $S \subset C^k$ be a bounded holomorphically convex domain and let $f\colon S \to C^n$ be a proper holomorphic map. Then $f(S)$ is an essentially transcendental set.

B. Let X be an irreducible almost-algebraic set in C^n and f be a bounded holomorphic function on X. Then f is constant.

Definition 1. Let X be an analytic subset of C^n, X is said to be *almost-algebraic* if and only if there exist: V-an affine algebraic subset of C^n and a biholomorphic map $g\colon X \to V$. If X is not almost-algebraic then it is said to be *essentially transcendental.*

An almost-algebraic set may not be algebraic. Let $X := \{x \in C^2\colon x_2 - e^{x_2} = 0\}$ and let $V := \{x \in C^2\colon x_2 = 0\}$. Applying some simple calculation we conclude that the set X is a determining set for polynomial functions on C^2.

Mappings: $\varphi\colon X \ni x \to (x_1, 0) \in V$ and $\psi\colon V \ni (x_1, 0) \to (x_1, e^{x_2}) \in X$ are holomorphic and bijective.

Hence X is transcendental and almost-algebraic. More generally, if X and Y are almost-algebraic sets and $g\colon X \to Y$ is a holomorphic map, then the graph of the map g is almost-algebraic.

From the viewpoint of analytic geometry almost-algebraic sets and essentially transcendental sets are different. Theorems A and B give some insight in this difference. Theorem A gives some qualitative information about the range of the pseudoconvex domain by a proper holomorphic map. Theorem B is a natural generalization of classical Liouville's Theorem.

To make it clear and convenient for the reader a definition of a branched covering and a few examples of coverings will be given. Also the theorem of Stoll-Andreotti on branched coverings will be formulated. As elementary consequences of this theorem very simple proofs of theorems P and Q will be given. Next we will prove theorems I and II which give theorems A and B as corollaries.

Definition 2. Let X and Y be complex spaces and let $\varphi\colon X \to Y$ be a light (i.e. having finite fibres) holomorphic mapping. Take $a \in X$. An open neighborhood U of a is said to be *φ-distinguished* if and only if $\overline{U}$ is a compact and $\{a\} = \varphi^{-1}(\varphi(a)) \cap \overline{U}$.

The distinguished neighborhoods of a form a base of neighborhoods of a.

Let X and Y be pure m-dimensional complex spaces and let Y be locally irreducible and $\varphi: X \to Y$ be a light holomorphic map. We associate with φ a function $v_\varphi: X \to N$ defined in the following manner: $: X \ni a \to v_\varphi(a) := \limsup_{x \to a} \# [\varphi^{-1}(\varphi(x)) \cap U]$ where U is a φ-distinguished neighborhood of a.

It can be proved that $v_\varphi(a)$ is a natural number independent of a choice of a φ-distinguished neighborhood of a.

Definition 3. A subset T of a complex space X is said to be thin if and only if every point $x \in T$ has an open neighborhood U containing an analytic subset S of U of co-dimension 1 such that $T \cap U \subset S$.

LEMMA 1. ([2]). *Let X and Y be pure m-dimensional complex spaces, let Y be a locally irreducible complex space and $\varphi: X \to Y$ be a light holomorphic map. Then, for every natural number p, the set $N_p := \{x \in X: v_\varphi(x) \geqslant p\}$ is a thin analytic set.*

Definition 4. Let X and Y be pure m-dimensional complex spaces and Y be irreducible and locally irreducible. Then a holomorphic map $\varphi: X \to Y$ is said to be a *branched covering* if and only if it is light and proper.

LEMMA 2. ([8]). *Let $\varphi: X \to Y$ be a branched covering. Then the map $n_{\varphi,X}: Y \ni y \to n_{\varphi,X}(y) := \sum_{x \in X} v_\varphi(x, y) \in N$, where*

$$v_\varphi(x, y) := \begin{cases} 0 & \text{if } \varphi(x) \neq y \\ v_\varphi(x) & \text{if } \varphi(x) = y \end{cases}$$

is constant. The number $s(\varphi) = n_{\varphi,X}(y)$ will be called a sheet number of φ.

Now some examples of branched coverings will be given.

E 1. $X := C^2$, $Y := \{y \in C^3: y_1^2 - y_2 y_3 = 0\}$ is an analytic subset of C^3 with the isolated singularity at the point $0 \in Y$.
The map $\varphi: X \ni x \to (x_1 x_2, x_1^2, x_2^2) \in Y$ is proper and light and the space Y is normal and irreducible. Hence φ is a 2-sheeted branched covering.

E 2. $X := C^2$, $Y := \{y \in C^3: y_1^2 + y_2^2 - y_3^2 = 0\}$ is an analytic subvariety of C^3 which has an isolated singularity at the point $0 \in Y$.

The map $\varphi: X \ni x \to (x_1^2 - x_2^2, 2x_1 x_2, x_1^2 + x_2^2) \in Y$ is proper and light and the space Y is normal and irreducible. Hence φ is a 2-sheeted branched covering.

E 3. $X = Y := C^n$. The map $\varphi: C^n \ni x \to (-q_1(x_1, \ldots, x_n), \ldots, (-1)^n q_n(x_1, \ldots, x_n)) \in C^n$ where q_ν denotes the ν-th symmetric polynomial of variables $x_1, \ldots, x_n$, is a $n!$-sheeted branched covering.

E 4. $X = Y := \{x \in C: |x| < 1\}$. Let $a_1, \ldots, a_m \in X$, $t \in R$. The map

$$\varphi: X \ni x \to e^{it} \prod_{\mu=1}^{m} \frac{x - a_\mu}{\bar{a}_\mu x - 1} \in X$$

is an m-sheeted branched covering.

In examples E 1, E 2 and E 3 it is easy to see that φ is proper because $|\varphi(x)| \to \infty$ when $|x| \to \infty$. In example E 4 the fact that φ is proper is proved in [6]. The map φ in examples E 1, E 2 and E 3 is light because, for every $y \in Y$, the analytic set $\varphi^{-1}(y)$ is finite.

THEOREM on branched coverings [2]. *Let $\varphi: X \to Y$ be an s-sheeted branched covering. Then*

1° *The map φ is open and surjective*

2° *The space X consists of finitely many branches $B_1, \ldots, B_r$, where $r \leqslant s$. Moreover, $\varphi(B_\lambda) = Y$ for $\lambda = 1, \ldots, r$.*

3° *If $\varphi^{-1}(\varphi(a)) = \{a\}$ for some point $a \in X$, then X is connected. If, in addition, X is locally irreducible at a, then X is irreducible.*

4° *The set $N_2 := \{x \in X: v_\varphi(x) > 1\}$ is thin and analytic in X. Moreover, $S' = \varphi(N_2)$ and $S = \varphi^{-1}(S')$ are also thin analytic subsets of Y and X, respectively.*

5° *The restriction $\varphi_0 := \varphi: X \setminus S \to Y \setminus S'$ is a locally topological, holomorphic, surjective, proper map. If Y is normal, then φ_0 is locally biholomorphic.*

6° *If $y \in Y \setminus S'$, then $\# \varphi^{-1}(y) = s$. If $y \in S'$, then $\# \varphi^{-1}(y) < s$.*

7° *If $s = 1$, then $\varphi: X \to Y$ is topological. If in addition Y is normal, then φ is biholomorphic.*

This theorem implies two corollaries.

THEOREM P: *Let X and Y be pure dimensional complex spaces, X is Stein, Y is locally irreducible and irreducible and let $\varphi: X \to Y$ be a proper holomorphic map. Then $\varphi: X \to Y$ is a branched covering.*

Proof. It suffices to show that the map $\varphi: X \to Y$ has discrete fibres. It is obvious, because for every $y \in Y$, $\varphi^{-1}(y)$ is a compact analytic subset of the Stein space X.

THEOREM Q: *Let U be a domain in $\mathbf{C}^n$ and let V be a pure k-dimensional analytic subvariety of U. Then, for each $x \in V$, there exist: Y-a k-dimensional linear subvariety of $\mathbf{C}^n$, N_k-a neighborhood of the point $\mathrm{pr}_Y(x)$ in Y, N_{n-k}-a neighborhood of the point $\mathrm{pr}_{Y^\perp}(x)$ in $Y^\perp$; where $Y^\perp$ is the orthogonal supplement of Y, pr_Y and $\mathrm{pr}_{Y^\perp}$ are linear projections of $\mathbf{C}^n$ on Y and on $Y^\perp$ respectively, such that the map*

$$\pi := \mathrm{pr}_Y | V \cap (N_k \times N_{n-k}): \ V \cap (N_k \times N_{n-k}) \to N_k$$

is a branched covering.

Proof. We can restrict our attention to the case in which U is a connected open neighborhood of $0 \in \mathbf{C}^n$.

We have two known lemmas:

LEMMA a ([9]). *Let U be a domain in $\mathbf{C}^n$ and let V be a pure k-dimensional analytic subvariety of U. Then $\mu_{2k+1}(V) = 0$, where μ_{2k+1} denotes $2k+1$-dimensional Hausdorff measure.*

LEMMA b ([9]). *Let U be a domain in $\mathbf{C}^n$, $0 \in U$ and let V be a closed subset of U such that $\mu_{2k+1}(V) = 0$. Then there exist: Y-a k-dimensional linear subvariety of $\mathbf{C}^n$, N_k-a neighborhood of the point 0 in Y, N_{n-k}-a neighborhood of the point 0 in $Y^{\perp}$, where $Y^{\perp}$ is the orthogonal supplement of Y, such that the map π, defined as in the hypothesis, is proper.*

Now applying theorem P to our situation, we conclude the proof of Theorem Q. It is convenient to introduce the following:

Definition 5. Let A and B be C-algebras. The homomorphism of C-algebras $h: B \to A$ is said to be *integral* if for each element $a \in A$ there exist a natural number n and elements $b_1, \ldots, b_n \in B$ such that $a^n + h(b_1)a^{n-1} + \ldots + h(b_n) = 0$.
An integral homomorphism h is called an *integral extension* if it is injective.

THEOREM I. *Let X and Y be complex irreducible spaces of dimension n. Let X be locally irreducible and Y be normal. Let $\varphi: X \to Y$ be a branched covering. If we denote by $H^{\infty}(X)$ and $H^{\infty}(Y)$ the $\mathbf{C}$-algebras of all bounded holomorphic functions on X and Y, respectively, then the map $H^{\infty}(Y) \to g \to \varphi^*(g) := g \circ \varphi \in H^{\infty}(X)$ is an integral extension.*

Proof. Let m denote a sheet number of φ. It is sufficent to show that for each $f \in H^{\infty}(X)$ we can construct a polynomial

$$P_f(T) := T^m + b_1 T^{m-1} + \ldots + b_m \in \varphi^*(H^{\infty}(Y))[T]$$

such that $P_f(f) = 0$.

Let $f \in H^{\infty}(X)$. For each $y \in Y \setminus S$ we have $\varphi^{-1}(y) = \{x_1 \ldots x_m\}$, where x_i depends on y for $i \in \{1, \ldots, m\}$. Let q_ν denote the ν-th symmetric polynomial of m independent variables. Then for every $\nu \in \{1, \ldots, m\}$, we can define the following function:

$$q_\nu f: Y \setminus S \ni y \to q_\nu(f(x_1), \ldots, f(x_m)) \in \mathbf{C}, \quad \text{where} \quad \{x_1, \ldots, x_m\} = \varphi^{-1}(y).$$

We shall show that $q_\nu f$ is analytic on $Y \setminus S$. For each $y \in Y \setminus S$ there exists a neighborhood V of y in $Y \setminus S$ such that the following conditions are satisfied:

1. $\varphi^{-1}(V) = U_1 \cup \ldots \cup U_m$
2. If $\alpha \neq \beta$ and $\alpha_1 \beta \in \{1, \ldots, m\}$, then $U_\alpha \cap U_\beta = \emptyset$
3. For each $j \in \{1, \ldots, m\}$, the set U_j is open in $X \setminus S$ and the map $\varphi_j := \varphi | U_j: U_j \to V$ is biholomorphic.

Observe that for every $z \in V$ we have the equality

$$(q_\nu f)(z) = q_\nu((f \circ \varphi_1^{-1})(z), \ldots, (f \circ \varphi_m^{-1})(z)).$$

From this we conclude that $(q_\nu f)|V$ is holomorphic on V. Thus $q_\nu f$ is holomorphic on $Y \setminus S$. Moreover, for every $\nu \in \{1, \ldots, m\}$, we have the inequality $|(q_\nu f)(y)| \leqslant q_\nu(M, \ldots, M)$, where $M = \sup\{|f(x)|: x \in X\}$. This means that $q_\nu f \in H^{\infty}(Y \setminus S)$. Observe that Y is a normal variety and S is a thin analytic subset of Y. Hence there exists a unique holomorphic extension of $q_\nu f$ to the whole variety Y, denoted also by $q_\nu f$.
Obviously, $q_\nu f \in H^{\infty}(Y)$.

Now we shall construct the required polynomial P_f. Let

$$P_f(T) := T^m - [(q_1 f)\circ\varphi]T^{m-1} + \ldots + (-1)^m[(q_m f)\circ\varphi].$$

It is sufficient to prove that $P_f(f) = 0$. Obviously $P_f(f) \in H^\infty(X)$. By elementary properties of symmetric polynomials and by the definition of $q_\nu f$ we have the following equality:

$$P_f(f)(x) = \prod_{1\leqslant\nu\leqslant m} [f(x) - f(x_\nu)]$$

for every $x \in X\setminus S$, where $\{x_1, \ldots, x_m\} = \varphi^{-1}(\varphi(x))$. Let $x \in X\setminus S$. Then $x \in \varphi^{-1}(\varphi(x))$ and $x_\nu = x$ for some $\nu \in \{1, \ldots, m\}$. We have

LEMMA ([1]). *Let X be any irreducible n-dimensional complex space and let $g: X \to C$ be a holomorphic function not identically zero in X. Then the analytic set $g^{-1}(0)$ is either empty or is pure $(n-1)$-dimensional or is nowhere dense in X.* Applying this Lemma to the function $g(x) := P_f(f)(x)$ in our situation we conclude that $P_f(f)(x) = 0$ for every $x \in X$.

THEOREM II. *Let X be a d-dimensional irreducible almost-algebraic subset of C^n. Then there exists a branched covering $\varphi: X \to C^d$.*

Proof. We may restrict ourselves to the case in which X is an algebraic d-dimensional subvariety of C^n. It is convenient to divide the proof into the following steps. First we present the Theorem II which is based on the Noether normalization lemma.

Next we formulate the necessary definition of algebraically proper map and the Theorem of Chevalley which gives the equivalence between integral homomorphisms and algebraically proper maps. Then we formulate the theorem of Grothendieck saying that every algebraically proper map is topologically proper.

THEOREM II. *Let X be an affine subvariety of C^n. Then there exists a regular map $\varphi: X \to C^d$ such that the following conditions are satisfied*

1. *φ is surjective;*
2. *$\# \varphi^{-1}(y) < \infty$ for every $y \in C^d$;*
3. *$\varphi^*: A(C^d) \to A(X)$ is an integral extension.*

Proof. Let $A(X)$ be a C-algebra of all regular functions on X and let $I(X)$ denote the ideal of all polynomials vanishing on X. Let $\varepsilon: C[T_1, \ldots, T_n] \to A(X)$ be an epimorphism of C-algebras such that $\operatorname{Ker}\varepsilon = I(X)$.

Let $F(X)$ be a finitely generated transcendental extension of the field C. Hence the field $F(X)$ has a finite transcendente degree over C.

Now we recall the well-known lemma which will be useful for our considerations.

LEMMA [3]. *Let A be a C-algebra such that there exists an epimorphism*

$$\varepsilon: C[T_1, \ldots, T_n] \to A \quad \textit{and} \quad \operatorname{Ker}\varepsilon$$

is a prime ideal. Let F be the field of fractions of A and let d be the transcendence degree of F over $\mathbf{C}$. *Then there exist* $\mu_1, \ldots, \mu_d \in A$ *such that* $\mu_1, \ldots, \mu_d$ *are algebraically independent over* $\mathbf{C}$ *and the inclusion map* $\mathbf{C}[\mu_1, \ldots, \mu_d] \to A$ *is an integral extension.*

Applying this Lemma to our situation we conclude that there exists an integral extension $h\colon \mathbf{C}[y_1, \ldots, y_d] \to A(X)$, where $\mathbf{C}[y_1, \ldots, y_d]$ and $A(\mathbf{C}^d)$ are isomorphic algebras.

Now we proceed to prove that there exists a unique regular map $\varphi\colon X \to \mathbf{C}^d$ such that $\varphi^* = h$, where $\varphi^*\colon A(\mathbf{C}^d) \ni f \to f \circ \varphi \in A(X)$.
Let $\pi_\nu\colon \mathbf{C}^d \ni x \to x_\nu \ni \mathbf{C}$, $1 \leqslant \nu \leqslant d$ be projections.

Then the map $\varphi\colon X \ni x \to \big(h(\pi_1), \ldots, h(\pi_d)\big)(x) \in \mathbf{C}^d$ is regular, surjective and has finite fibres (see [7] for details).

We use the following THEOREM of Chevalley [5]. *Let X and Y be affine varieties and let* $\varphi\colon X \to Y$ *be a regular map. Then the following conditions are equivalent:*

1. $\varphi^*\colon A(Y) \to A(X)$ *is an integral homomorphism of the algebras of regular functions;*
2. $\varphi\colon X \to Y$ *is algebraically proper, i.e. the mapping*

$$\varphi \times 1_Z\colon X \times Z \ni (x, z) \to (\varphi(x), z) \in Y \times Z$$

is closed in the sense of Zariski for every quasi-projective variety Z, where $X \times Z$ *and* $Y \times Z$ *denote products in the sense of Zariski.*

Therefore, the mapping φ is algebraically proper. It is also topologically proper by the following

THEOREM of Grothendieck [4]. *Let X and Y be a quasi-projective varieties and let* $\varphi\colon X \to Y$ *be a regular algebraically proper map. Then* φ *is proper.*

Thus the proof of Theorem II is finished. As corollaries to theorems I and II we have

THEOREM A. *Let S be a Stein variety which admits a non-constant bounded holomorphic function. Let V be an affine algebraic variety and let* $\psi\colon S \to V$ *be a proper holomorphic map. Then* $\psi(S)$ *is essentially transcendental.*

Proof. Suppose that $\psi(S)$ is almost-algebraic. Then, by Theorem II there exist a linear variety Y and a branched covering map $\varphi\colon \psi(S) \to Y$.
Applying Theorem I to this covering map we obtain the injective integral homomorphism $\varphi^*\colon H^\infty(Y) \ni f \to f \circ \varphi \in H^\infty(\psi(S))$.

Observe, that by the theorem of Liouville the $\mathbf{C}$-algebra $H^\infty(Y)$ is isomorphic to $\mathbf{C}$. Now, because $\varphi^*(H^\infty(Y))$ is a field, then $H^\infty(\psi(S))$ is an algebraic extension of the field $\varphi^*(H^\infty(Y))$. Following the fact that $\varphi^*(H^\infty(Y))$ is isomorphic to $\mathbf{C}$ we conclude that $H^\infty(\psi(S))$ is isomorphic to $\mathbf{C}$. This contradicts the assumption that there exists on S a non-constant bounded holomorphic function.

Analogously we can obtain

THEOREM B. *Let X be an almost-algebraic variety. Then every bounded holomorphic function on X is constant.*

Remark. Another proof of the Theorem B has been given independently by W. Stoll in "Value distribution theory on parabolic spaces" Lecture Notes in Math. 600, Springer 1977.

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