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Systems of Difference Inequalities of the Elliptic Type

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§ **1.** This paper is concerned with the systems I and II of difference inequalities of the elliptic type, cf. the formula (4.8) and (4.9).

The suitable assumptions being involved we shall be able to obtain estimates on the solution r_l^M $(l = 1, \dots, p)$ of these difference inequalities that assure its convergence to zero in the limit as the mesh size h approaches zero, cf. Theorem 4 and Theorem 5.

Three estimates of that kind will be derived, cf. Theorem 6 (§ 24), Theorem 7 (§ 27) and Remark 7 (§ 27).

These results can be used to establish the convergence and the error estimate of the difference method for systems of elliptic differential equations. Denoting by u^M an approximate solution of a difference equation at the nodal point x^M, we can obtain from Theorem 6 the effective estimate for $u^M - u(x^M)$, $u(x)$ being the solution of the corresponding system of differential equations. The results will be published later.

For really computing the solution of a particular boundary problem the estimations (5.2) can easily be obtained. However, for the theoretical proof of convergence of the difference method for systems of elliptic equations the difficulty of proving the conditions (5.2) should not be overlooked also in this paper, cf. A. Pliś [2], and [4]. We shall assume here that conditions (5.2) are satisfied.

§ **2.** Let us denote by Q the set of points $x \in R^n$, $x = (x_1, \dots, x_n)$:

$$Q\colon 0 \leqslant x_j \leqslant \sigma \quad (j = 1, \dots, n),\ 0 < \sigma = \text{const}. \tag{2.1}$$

Let us denote by M the sequence of indices

$$M = (m_1, m_2, \dots, m_n),\ 0 \leqslant m_j \leqslant N \quad (j = 1, 2, \dots, n), \tag{2.2}$$

and by x^M the nodal point with coordinates

$$x^M = (x_1^M, x_2^M, \dots, x_n^M), \tag{2.3}$$

where $x_j^M = m_j \cdot h$ $(j = 1, \dots, n)$, $0 < h = \sigma/N$, N being the natural number.

We shall introduce also the nodal points in the set Q characterized by the following sequences of indices:

$$(2.4)\qquad \begin{cases} j(M) = (m'_1, \dots, m'_n),\ m'_j = m_j+1,\ m'_i = m_i & \text{for } i \neq j, \\ -j(M) = (m'_1, \dots, m'_n),\ m'_j = m_j-1,\ m'_i = m_i & \text{for } i \neq j, \\ (i = 1, \dots, n;\ j = 1, \dots, n), \end{cases}$$

and for $i \neq j$:

$$(2.5)\qquad \begin{cases} ij(M) = (m'_1, \dots, m'_n),\ m'_i = m_i+1,\ m'_j = m_j+1, \\ -ij(M) = (m'_1, \dots, m'_n),\ m'_i = m_i-1,\ m'_j = m_j+1, \\ -i-j(M) = (m'_1, \dots, m'_n),\ m'_i = m_i-1,\ m'_j = m_j-1, \\ i-j(M) = (m'_1, \dots, m'_n),\ m'_i = m_i+1,\ m'_j = m_j-1, \end{cases}$$

where $m'_s = m_s$ in the formula (2.5) for $s = 1, \dots, n;\ s \neq i,\ s \neq j$, cf. Fig. 1.

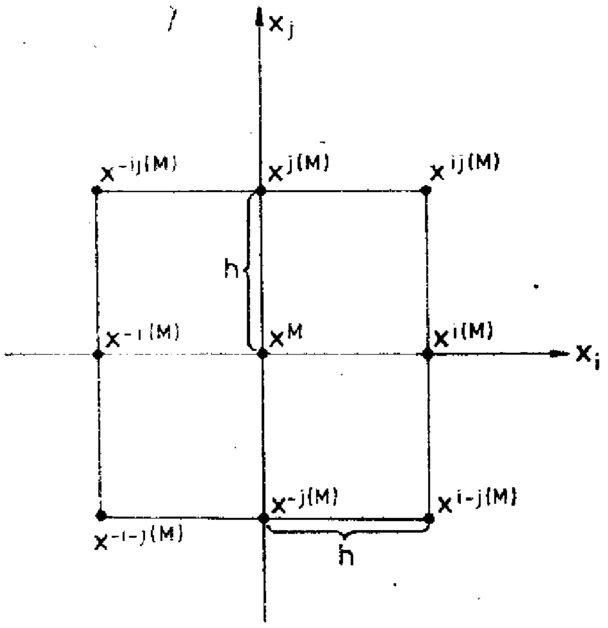

Fig. 1. The nodal points x^M, $x^{i(M)}$, $x^{ij(M)}$, ... For the sake of simplicity the nodal point x^M has been located at the origin

The nodal point $x^{ij(M)}$ can be denoted also by $x^{ji(M)}$, since we define

$$(2.6)\qquad \begin{cases} ij(M) = ji(M),\ -ij(M) = j-i(M),\ -i-j(M) = -j-i(M), \\ i-j(M) = -ji(M), \quad \text{for } i \neq j\ (i,j = 1, \dots, n). \end{cases}$$

We denote by $\operatorname{int} Q$ the set of nodal points (2.3) which belong to the interior of the set Q, cf. (2.1), and by sym A the set of nodal points x^M such that $x^M \in \operatorname{int} Q$ and $x^{M^*} \in \operatorname{int} Q$ simultaneously, x^{M^*} and x^M being symmetric with respect to the nodal point x^A.

§ **3.** Let us denote by r_l^M $(l = 1, 2, \dots, p)$ the value of the function r_l $(l = 1, \dots, p)$ at the nodal point x^M.

We shall use the difference quotients

$$(3.1)\qquad \begin{cases} r_{l+}^{Mj} = \frac{1}{h}\cdot(r_l^{j(M)} - r_l^M),\ r_{l-}^{Mj} = \frac{1}{h}\cdot(r_l^M - r_l^{-j(M)}), \\ (l = 1, \dots, p;\ j = 1, \dots, n), \end{cases}$$

$$(3.2)\qquad r_l^{Mj} = \frac{1}{2h}\cdot(r_l^{j(M)} - r_l^{-j(M)}),\ (l = 1, \dots, p;\ j = 1, \dots, n),$$

for the first partial derivatives, and the difference quotients

$$(3.3)\qquad \begin{cases} r_l^{Mjj} = h^{-2}\cdot(r_l^{j(M)} - 2\cdot r_l^M + r_l^{-j(M)}), \\ r_l^{Mij} = \frac{1}{4}\cdot h^{-2}(r_l^{ij(M)} - r_l^{-ij(M)} - r_l^{i-j(M)} + r_l^{-i-j(M)}), \\ (i \neq j;\ i = 1, \dots, n;\ j = 1, \dots, n;\ l = 1, \dots, p), \end{cases}$$

for the second derivatives.

From the definitions (3.1) (3.2) (3.3) it follows that

$$(3.4)\qquad \begin{cases} r_l^{Mj} = \frac{1}{2}\cdot(r_{l+}^{Mj} + r_{l-}^{Mj}),\ r_l^{Mjj} = \frac{1}{h}\cdot(r_{l+}^{Mj} - r_{l-}^{Mj}), \\ (j = 1, \dots, n;\ l = 1, \dots, p). \end{cases}$$

We shall use also the difference quotients r_{l++}^{Mij}, r_{l-+}^{Mij}, r_{l--}^{Mij}, r_{l+-}^{Mij}

$$(l = 1, \dots, p;\ i \neq j;\ i = 1, \dots, n;\ j = 1, \dots 1, \dots, n),$$

cf. Fig. 1:

$$(3.5)\qquad \begin{cases} r_{l++}^{Mij} = h^{-2}\cdot(r_l^{ij(M)} - r_l^{j(M)} - r_l^{i(M)} + r_l^M), \\ r_{l-+}^{Mij} = h^{-2}\cdot(r_l^{j(M)} - r_l^{-ij(M)} - r_l^M + r_l^{-i(M)}), \\ r_{l--}^{Mij} = h^{-2}\cdot(r_l^M - r_l^{-i(M)} - r_l^{-j(M)} + r_l^{-i-j(M)}), \\ r_{l+-}^{Mij} = h^{-2}\cdot(r_l^{i(M)} - r_l^M - r_l^{i-j(M)} + r_l^{-j(M)}). \end{cases}$$

From the definitions (3.5) and (3.3) we obtain

$$(3.6)\qquad \begin{cases} r_l^{Mij} = \frac{1}{4}\cdot(r_{l++}^{Mij} + r_{l-+}^{Mij} + r_{l--}^{Mij} + r_{l+-}^{Mij}), \\ (l = 1, \dots, p;\ i = 1, \dots, n;\ j = 1, \dots, n;\ i \neq j). \end{cases}$$

§ **4.** Let us consider the following conditions W_j $(j = 1, 2, 3)$:

Condition W$_1$. The quadratic forms

$$(4.1)\qquad \sum_{i,j=1}^{n} a_{lij}^M\cdot\lambda_i\cdot\lambda_j \qquad (l = 1, \dots, p)(x^M \in \operatorname{int} Q),$$

are positive definite and the characteristic roots s_{lj}^M $(l = 1, \dots, p;\ j = 1, \dots, n)$, $s_{lj}^M > 0$, are bounded:

$$(4.2)\qquad 0 < \delta_1 \leqslant s_{lj}^M \leqslant \delta_2 \qquad (l = 1, \dots, p;\ j = 1, \dots, n),$$

the constants δ_1 and δ_2 being independent of the mesh size h.

Condition W_2. The elements of the matrix (c_{lk}^M) $(l = 1, \dots, p;\ k = 1, \dots, p)$ $(x^M \in \operatorname{int} Q)$ satisfy the inequalities

$$c_{ll}^M \leqslant \eta < 0 \quad (l = 1, \dots, p;\ \eta = \text{const}), \tag{4.3}$$

$$0 \leqslant c_{lk}^M < \delta \quad (\delta = \text{const},\ l \neq k;\ l = 1, \dots, p;\ k = 1, \dots, p), \tag{4.4}$$

where the constants η and δ does not depend on the mesh size h and

$$-\frac{1}{p-1} < \gamma < 0 \quad (p \geqslant 2), \tag{4.5}$$

the coefficient γ being defined by

$$\gamma = +\eta^{-1} \cdot \delta \quad (\gamma < 0). \tag{4.6}$$

Condition W_3. The coefficients a_{lij}^M, b_{lj}^M are bounded:

$$|a_{lij}^M| \leqslant \zeta, \quad |b_{lj}^M| \leqslant \beta, \tag{4.7}$$

for $l = 1, \dots, p$; $i = 1, \dots, n$; $j = 1, \dots, n$; $x^M \in \operatorname{int} Q$, the constants ζ and β being independent of the mesh size h.

Let us consider two systems I and II of difference inequalities for the functions r_l^M $(l = 1, \dots, p)$, defined at the nodal points x^M:

System I:

$$\sum_{i,j=1}^{n} a_{lij}^M \cdot r_l^{Mij} + \sum_{j=1}^{n} b_{lj}^M \cdot r_l^{Mj} + \sum_{k=1}^{p} c_{lk}^M \cdot r_k^M \geqslant -\varepsilon(h), \tag{4.8}$$

where $0 < \varepsilon(h) = \text{const}$ $(l = 1, \dots, p)$.

System II:

$$\sum_{i,j=1}^{n} a_{lij}^M \cdot r_l^{Mij} + \sum_{j=1}^{n} b_{lj}^M \cdot r_l^{Mj} + \sum_{k=1}^{p} c_{lk}^M \cdot r_k^M \leqslant +\varepsilon(h), \tag{4.9}$$

where $0 < \varepsilon(h) = \text{const}$ $(l = 1, \dots, p)$.

We shall use the following definition:

DEFINITION 1. We say that the systems I and II are of the *elliptic type* if the conditions W_1, W_2 and W_3 are fulfilled.

§ 5. Let us summarize now the principal assumptions:

ASSUMPTIONS H. We shall assume that 1° the functions r_l^M $(l = 1, \dots, p)$ are defined at the nodal points (2.3) of the set Q, $x^M \in Q$, cf. (2.1).

2° There exists a positive constant $\vartheta>0$ (independent of the mesh size h) such that the first order difference quotients satisfy the conditions

$$(5.1)\qquad \begin{cases} |r^{Mj}_{l+}|\leqslant h\cdot\vartheta, & \text{for } x^M\in\partial Q,\ x^{j(M)}\in \operatorname{int} Q\ (j=1,\dots,n), \\ |r^{Mij}_{l-}|\leqslant h\cdot\vartheta, & \text{for } x^M\in\partial Q,\ x^{-j(M)}\in \operatorname{int} Q\ (j=1,\dots,n), \\ (l=1,\dots,p), \end{cases}$$

at the nodal points x^M on the boundary ∂Q of the set Q.

3° There exists a positive constant $L>0$ (independent of the mesh size h) such that the second order difference quotients satisfy the conditions

$$(5.2)\qquad \begin{cases} |r^{Mjj}_{l}-r^{Pjj}_{l}|\leqslant h\cdot L,\ |r^{Mij}_{l++}-r^{Pij}_{l++}|\leqslant h\cdot L, \\ |r^{Mij}_{l-+}-r^{Pij}_{l-+}|\leqslant h\cdot L,\ |r^{Mij}_{l--}-r^{Pij}_{l--}|\leqslant h\cdot L, \\ |r^{Mij}_{l+-}-r^{Pij}_{l+-}|\leqslant h\cdot L \quad (i\neq j;\ l=1,\dots,p) \end{cases}$$

at the nodal points x^M and x^P, $P=s(M)$ $(s=\pm1,\pm2,\dots,\pm n)$, the distance between x^M and x^P being h in the direction of the x_s-axis.

4° We suppose that

$$(5.3)\qquad |r^{Mij}_{l}|\leqslant \Lambda\ \ (i=1,\dots,n;\ j=1,\dots,n;\ l=1,\dots,p)\ \ (x^M\in\operatorname{int} Q),$$

where the constant Λ is independent of the mesh size h.

5° We suppose also that r^M_l takes on the prescribed values:

$$(5.4)\qquad r^M_l=0, \quad \text{for } x^M\in\partial Q\ \ (l=1,\dots,p),$$

at the nodal points x^M on the boundary ∂Q of the set Q.

6° We suppose finally that the systems I and II of difference inequalities are of the elliptic type, cf. Definition 1 (§ 4).

§ 6. We shall formulate now Lemma 1 on the location of the functions r^M_l $(l=1,\dots,p)$ for $x^M\in Q$.

LEMMA 1. *Let us suppose that the functions* r^M_l $(l=1,\dots,p)$ *satisfy part* 1°, 2°, 4° *and* 5° *of the assumptions* H, *cf.* § 5, *and let us consider the function*

$$(6.1)\qquad z=(\tfrac{1}{2}\cdot\Lambda+\vartheta)\cdot y^2,\ \ 0\leqslant y<+\infty.$$

Under these assumptions we have

$$(6.2)\qquad |r^M_l|\leqslant(\tfrac{1}{2}\cdot\Lambda+\vartheta)\cdot y^2\ \ (l=1,\dots,p),$$

where y *denotes the Euclidean distance* $\varrho(x^M,\partial Q)$ *of the point* $x^M\in Q$ *from the boundary* ∂Q, $y=\varrho(x^M,\partial Q)$. *In the formula* (6.2) *we have the strong inequality for* $\varrho(x^M,\partial Q)>0$, *cf.* Fig. 2.

The proof of Lemma 1 can be found in the previous paper, cf. [3], § 12 (it is sufficient to repeat the proof of Lemma 1 of the paper [3] for each function r^M_l $(l=1,\dots,p)$, successively).

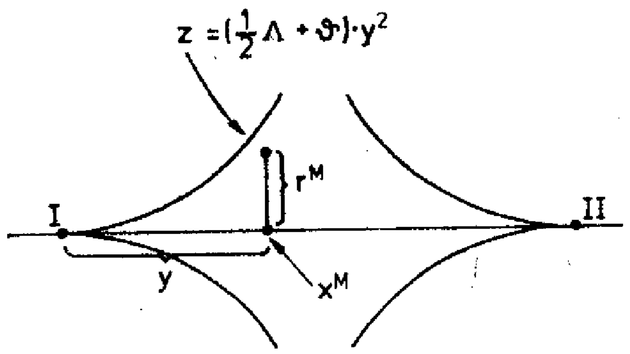

Fig. 2. The location of the function r_l^M $(l = 1, \dots, p)$. The set $Q = (\mathrm{I}, \mathrm{II})$ as seen from the edge in the case $n = 2$

§ 7. **Notations.** Let us denote

$$r_l^{A_l} = \max_{x^M \in Q} r_l^M,\ A_l = A_l(h),\ A_l = (a_{l1}, \dots, a_{ln}),\ x^{A_l} \in \operatorname{int} Q, \tag{7.1}$$

$$r_l^{B_l} = \max_{x^M \in Q} r_l^M,\ B_l = B_l(h),\ B_l = (b_{l1}, \dots, b_{ln}),\ x^{B_l} \in \operatorname{int} Q, \tag{7.2}$$

for $l = 1, \dots, p$. Let us denote in addition, cf. [3] (§ 13):

$$\varkappa = (\tfrac{1}{2}\cdot \Lambda + \vartheta)^{-1/2}. \tag{7.3}$$

$$|m - a_l| = \sum_{j=1}^{n} |m_j - a_{lj}| \quad (l = 1, \dots, p). \tag{7.4}$$

We shall consider the sets V^{A_l} and $\operatorname{supp} V^{A_l}$ (support of the set V^{A_l}) for $l = 1, \dots, p$, connected with the nodal point x^{A_l} $(l = 1, \dots, p)$:

$$V^{A_l} = \{x\colon |x_j - x_j^M| \leqslant h,\ (j = 1, \dots, n),\ h\cdot|m - a_l| \leqslant \varkappa h^\alpha\}, \tag{7.5}$$

$$\operatorname{supp} V^{A_l} = \{x^M\colon h\cdot|m - a_l| \leqslant \varkappa h^\alpha\}, \tag{7.6}$$

where $\frac{1}{2} \leqslant \alpha < 1$ $(l = 1, \dots, p)$.

In a similar way for x^{B_l} $(l = 1, \dots, p)$ we define

$$V^{B_l} = \{x\colon |x_j - x_j^M| \leqslant h\ (j = 1, \dots, n),\ h\cdot|m - b_l| \leqslant \varkappa h^\alpha\}, \tag{7.7}$$

$$\operatorname{supp} V^{B_l} = \{x^M\colon h\cdot|m - b_l| \leqslant \varkappa h^\alpha\}. \tag{7.8}$$

By $\varrho(x^{A_l}, \partial Q)$ we shall denote the euclidean distance of the nodal point x^{A_l} from the boundary ∂Q of the set Q, and by $\varrho(x^{A_l}, x^M)$ the Euclidean distance between the nodal points x^{A_l} and x^M.

§ 8. **Lemma 2.** *Let us suppose that the functions* r_l^M *(*$l = 1, \dots, p$ *satisfy parts* 1°, 2°, 4° *and* 5° *of the assumptions* H, cf. § 5.

We suppose also that the inequalities

$$r_l^{A_l} \geqslant h,\ r_l^{B_l} \leqslant -h, \quad \textit{for } h > 0,\ A_l = A_l(h),\ B_l = B_l(h), \tag{8.1}$$

hold for some l $(l = 1, \dots, p)$, cf. *Notations* § 7.

Under these assumptions we have

$$\sup V^{A_l} \subset Q, \ \operatorname{supp} V^{B_l} \subset Q, \tag{8.2}$$

$$|x^{b_l} - x^{A_l}| \geqslant \varkappa h^{\alpha}, \ |x^{b'_l} - x^{B_l}| \geqslant \varkappa h^{\alpha}, \tag{8.3}$$

where $x^{b_l} \in \partial V^{A_l}$, $x^{b'_l} \in \partial V^{B_l}$, $\frac{1}{2} \leqslant \alpha < 1$, $0 < h < 1$, *cf. Fig.* 3 *and Fig.* 4.

The proof of Lemma 2 can be found in [3], (cf. [3], § 14).

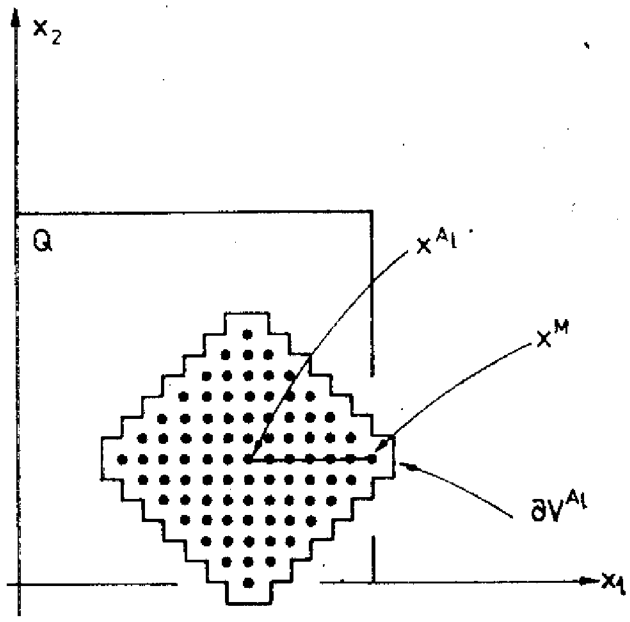

Fig. 3. The sets V^{A_l} and supp V^{A_l} $(l = 1, \ldots, p)$ in the two dimensional case $n = 2$. In this figure we have supp $V^{A_l} \subset Q$

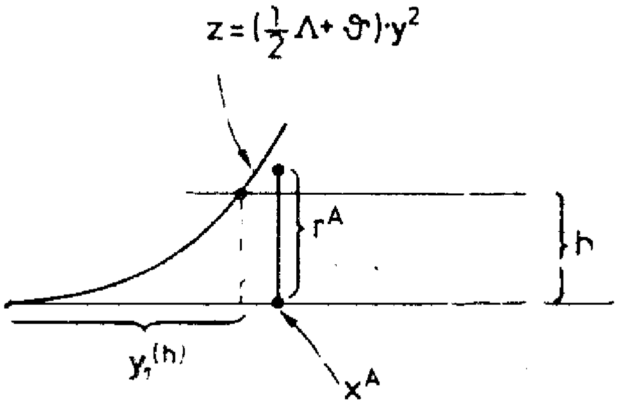

Fig. 4. $y_l(h)$ and the center x^{A_l} of the set supp V^{A_l} $(l = 1, \ldots, p)$

§ 9. **Remark 1.** The difference inequality (4.8) can be written in the form

$$\sum_{i,j=1}^{n} a_{lij}^{M} \cdot r_l^{Mij} + \sum_{j=1}^{n} b_{lj}^{M} \cdot r_l^{Mj} + c_{ll}^{M} \cdot r_l^{M} \geqslant -\varepsilon(h) - \sum_{k=1, k \neq l}^{p} c_{lk}^{M} \cdot r_k^{M}, \tag{9.1}$$

the left hand member being dependent on the function r_l^M and of the difference quotients r_l^{Mij}, r_l^{Mj}, only. The right hand side contains the remaining functions r_k^M for $k \neq l$.

Hence, (9.1) can be regarded as the difference inequality for one function r_l^M, only, the right hand side being considered for a moment as a free member.

Then, in a similar way as in [1] (cf. [1], Theorem 1, § 11) the estimate for the maximum value $r_l^{A_l}$ can be obtained. That estimate will be given in § 12.

§ **10.** To achieve the estimate for $r_l^{A_l}$, as explained in § 9, the results and notations of Lemma 3, Remark 2 and Remark 3 will be needed, all of them being placed in the next paragraph, cf. § 11. But the introduced notions are so similar to that used in [1] that we have decided to accept all the notations of the paper [1] (with minor changes) in order to facilitate the comparison between the theory for one difference inequality of the elliptic type and the corresponding theory for a system of difference inequalities.

§ **11.** LEMMA 3. *Let us suppose that the functions* r_l^M $(l = 1, \ldots, p)$ *satisfy Assumptions* H (cf. § 5 *and Notations* § 7).

Let us denote by $S_{A_l}(x)$, $S_{B_l}(x)$, $x = (x_1, \ldots, x_n) \in R^n$ *the quadratic forms*

$$(11.1)\qquad \begin{cases} S_{A_l}(x) = \frac{1}{2}\cdot \sum\limits_{i,j=1}^{n} r_l^{A_l ij}\cdot(x_i - x_l^{A_l})(x_j - x_j^{A_l}), & (x \in R^n), \\ S_{B_l}(x) = \frac{1}{2}\cdot \sum\limits_{i,j=1}^{n} r_l^{B_l ij}\cdot(x_i - x_i^{B_l})(x_j - x_j^{B_l}), & (x \in R^n) \end{cases}$$

and by $S_{A_l}^M$, $S_{B_l}^M$ *the values of* $S_{A_l}(x)$, $S_{B_l}(x)$ *at the nodal point* $x^M \in Q$, *respectively*:

$$(11.2)\qquad \begin{cases} S_{A_l}^M = \frac{1}{2} \sum\limits_{i,j=1}^{n} r_l^{A_l ij}\cdot(m_i - a_{li})(m_j - a_{lj})\cdot h^2, & (x^M \in Q), \\ S_{B_l}^M = \frac{1}{2}\cdot \sum\limits_{i,j=1}^{n} r_l^{B_l ij}\cdot(m_i - b_{li})(m_j - b_{lj})\cdot h^2, & (x^M \in Q). \end{cases}$$

Under these assumptions we have the following estimate for the quadratic forms $S_{A_l}(x)$, $S_{B_l}(x)$ *in the sets* V^{A_l} *and* V^{B_l} $(l = 1, \ldots, p)$, *respectively*:

$$(11.3)\qquad \begin{cases} S_{A_l}(x) \leqslant C_1(h), & \text{for } x \in V^{A_l}, \\ S_{B_l}(x) \geqslant -C_1(h), & \text{for } x \in V^{B_l}, \end{cases}$$

where $l = 1, \ldots, p$, *and*

$$(11.4)\qquad C_1(h) = 2\cdot|\theta|\cdot L\varkappa^3 h^{3\alpha} + n\cdot 2\Lambda\cdot\varkappa h^{1+\alpha} + n^2\cdot\Lambda\cdot h^2,\ |\theta| < 1.$$

The proof of Lemma 3 will be omitted, since it is similar to the proof of Lemma 5 in the paper [1].

Remark 2. The quadratic forms $S_{A_l}(x)$ and $S_{B_l}(x)$ can be estimated for $x \in R^n$. For this purpose let us denote

$$(11.5)\qquad G_1(x - x^{A_l}, h) = \begin{cases} C_1(h), & \text{for } x \in V^{A_l}, \\ \lambda^2\cdot C_1(h), & \text{for } x \in R^n \setminus V^{A_l}, \end{cases}$$

where $x - x^{A_l} = \lambda\cdot(x^{b_l} - x^{A_l})$ $(\lambda \geqslant 1)$, x^{b_l} being the intersection point of the boundary ∂V^{A_l} with a segment joining the points x^{A_l} and x $(l = 1, \ldots, p)$.

From the fact that $S_{A_l}(x)$ is the quadratic form and from the estimate (11.3) it follows that

$$(11.6)\qquad S_{A_l}(x) \leqslant G_1(x - x^{A_l}, h), \quad \text{for } x \in R^n \ (l = 1, \ldots, p).$$

Let us denote also

$$G_1(x-x^{B_l}, h) = \begin{cases} C_1(h), & \text{for } x \in V^{B_l}, \\ \lambda'^2 \cdot C_1(h), & \text{for } x \in R^n \setminus V^{B_l}, \end{cases} \tag{11.7}$$

where $x-x^{B_l} = \lambda' \cdot (x^{b_l'} - x^{B_l})$ $(\lambda' \geqslant 1)$, $x^{b_l'}$ being the intersection point of the boundary ∂V^{B_l} with the segment joining the points x^{B_l} and x $(l = 1, \dots, p)$.

Then we have

$$S_{B_l}(x) \geqslant -G_1(x-x^{B_l}, h), \quad \text{for } x \in R^n. \tag{11.8}$$

Remark 3. Let us consider the expressions

$$\sum_{i,j=1}^{n} a_{lij}^{A_l} \cdot r_l^{A_l ij}, \quad \sum_{i,j=1}^{n} a_{lij}^{B_l} \cdot r_l^{B_l ij} \quad (l = 1, \dots, p), \tag{11.9}$$

the coefficients $a_{lij}^{A_l}$, $a_{lij}^{B_l}$ being taken from the quadratic form $\sum_{i,j=1}^{n} a_{lij}^{M} \cdot \lambda_i \lambda_j$, cf. § 4, the formula (4.1).

We shall verify that the expressions (11.9) have the following estimates:

$$\sum_{i,j=1}^{n} a_{lij}^{A_l} \cdot r_l^{A_l ij} \leqslant E^{A_l}(h), \quad \sum_{i,j=1}^{n} a_{lij}^{B_l} \cdot r_l^{B_l ij} \geqslant -E^{B_l}(h), \tag{11.10}$$

where

$$\begin{cases} E^{A_l}(h) = \sum_{k=1}^{n} G_1\big((s_{lk}^{A_l})^{1/2} \cdot \alpha_{lk}^{A_l};\ h\big), \\ E^{B_l}(h) = \sum_{k=1}^{n} G_1\big((s_{lk}^{B_l})^{1/2} \cdot \alpha_{lk}^{B_l};\ h\big), \end{cases} \tag{11.11}$$

and the functions $E^{A_l}(h)$ and $E^{B_l}(h)$ $(l = 1, \dots, p)$ are positives $E^{A_l}(h) > 0$, $E^{B_l}(h) > 0$, for $h > 0$.

In the formula (11.11) $s_{l1}^{A_l}, s_{l2}^{A_l}, \dots, s_{ln}^{A_l}$ denote positive characteristic roots of the form $f_1 = \sum_{i,j=1}^{n} a_{lij}^{A_l} \cdot \lambda_i \lambda_j$, cf. Condition W$_1$ (§ 4), $(\alpha_{lkj}^{A_l})$ $(k, j = 1, \dots, n)$ is the orthogonal matrix transforming f_1 to the canonical form, and $\alpha_{lk}^{A_l}$ denotes the unit vector

$$\alpha_{lk}^{A_l} = (\alpha_{lk1}^{A_l}, \alpha_{lk2}^{A_l}, \dots, \alpha_{lkn}^{A_l}) \ (k = 1, \dots, n).$$

We introduce similar notations for the form $\sum_{i,j=1}^{n} a_{lij}^{B_l} \cdot \lambda_i \lambda_j$.

In order to deduce the first formula (11.10) it is sufficient to observe that f_1 is the positive definite form, cf. Condition W$_1$ (§ 4), and the form $S_{A_l}(x)$:

$$S_{A_l}(x) = \sum_{i,j=1}^{n} r_l^{A_l ij} \cdot (x_i - x_i^{A_l})(x_j - x_j^{A_l}) = \tag{11.12}$$

$$= \sum_{i,j=1}^{n} r_l^{A_l ij} \cdot \mu_i \mu_j, \quad \text{for } \mu_i = x_i - x_i^{A_l} \ (i = 1, 2, \dots, n),$$

possesses the estimate (11.6), hence from the known lemma on quadratic forms, cf. Lemma 3 in the paper [1], we obtain the first part of (11.10).

The proof of the second part of (11.10) is similar.

§ 12. Now we shall give the estimate for the maximum value $r_l^{A_l}$ and the minimum value $r_l^{B_l}$.

Lemma 4. *Let us suppose that the functions* r_l^M $(l = 1, \ldots, p)$ *satisfy the Assumptions* H. *We shall use the notations of* § 7.

Let us suppose in addition that

$$\sum_{i,j=1}^{n} a_{lij}^{M} \cdot r_l^{Mij} + \sum_{j=1}^{n} b_{lj}^{M} \cdot r_l^{Mj} + \sum_{k=1}^{p} c_{lk}^{M} \cdot r_k^{M} \geqslant -\varepsilon(h), \tag{12.1}$$

for $\varepsilon(h) > 0$, *if* $r_l^M > 0$ $(l = 1, \ldots, p)$ *at the nodal point* x^M, $x^M \in \operatorname{int} Q$, *and*

$$\sum_{i,j=1}^{n} a_{lij}^{M} \cdot r_l^{Mij} + \sum_{j=1}^{n} b_{lj}^{M} \cdot r_l^{Mj} + \sum_{k=1}^{p} c_{lk}^{M} \cdot r_k^{M} \leqslant +\varepsilon(h), \tag{12.2}$$

if $r_l^M < 0$ $(l = 1, \ldots, p)$ *at the nodal point* x^M, $x^M \in \operatorname{int} Q$.

Under these assumptions we have

$$\left\{\begin{aligned} & r_l^{A_l} \leqslant g^{A_l}(h) - \eta^{-1} \cdot \sum_{k=1, k \neq l}^{p} c_{lk}^{A_l} \cdot r_k^{A_l}, \\ & r_l^{B_l} \geqslant -g^{B_l}(h) - \eta^{-1} \cdot \sum_{k=1, k \neq l}^{p} c_{lk}^{B_l} \cdot r_k^{B_l}, \end{aligned}\right. \tag{12.3}$$

where

$$\left\{\begin{aligned} 0 < g^{A_l}(h) &= -\eta^{-1} \cdot [E^{A_l}(h) + D(h) + \varepsilon(h)], \\ 0 < g^{B_l}(h) &= -\eta^{-1} \cdot [E^{B_l}(h) + D(h) + \varepsilon(h)]. \end{aligned}\right. \tag{12.4}$$

In the formula (12.4) *the quantities* $E^{A_l}(h)$, $E^{B_l}(h)$ *are defined by* (11.11), *and* $0 < D(h) = n\beta hA$.

Proof. Assuming the contrary, suppose that

$$r_l^{A_l} > g^{A_l}(h) - \eta^{-1} \cdot \sum_{k=1, k \neq l}^{p} c_{lk}^{A_l} \cdot r_k^{A_l}. \tag{12.5}$$

With (12.5) in mind we shall verify that

$$\sum_{i,j=1}^{n} a_{lij}^{A_l} \cdot r_l^{A_lij} \leqslant E^{A_l}(h), \tag{12.6}$$

$$\sum_{j=1}^{n} b_{lj}^{A_l} \cdot r_l^{A_lj} \leqslant D(h), \tag{12.7}$$

$$c_{ll}^{A_l} \cdot r_l^{A_l} \leqslant \eta \cdot r_l^{A_l}. \tag{12.8}$$

In fact, (12,6) follows from Remark 3, cf. (11.10).

The difference quotients at the nodal point x^{A_l} satisfy the inequalities $|r_l^{A_lj}|\leqslant h$, $|r_l^{A_lij}|\leqslant h\cdot\Lambda$, since x^{A_l} is the nodal point where the maximum value is attained, cf. Lemma 2 in the paper [1] (the formula (5.3)). Therefore, from (4.7) it follows that

$$\sum_{j=1}^{n} b_{lj}^{A_l}\cdot r_l^{A_lj}\leqslant n\cdot\beta\cdot h\cdot\Lambda = D(h)\,, \tag{12.9}$$

which completes the proof of (12.7)

Inequality (12.8) follows immediately from the assumption (4.3): $c_{ll}^{A_l}\leqslant\eta<0$, since $r_l^{A_l}\geqslant 0$ because of (5.4).

From (12.6) (12.7) (12.8) we obtain first by summation

$$\begin{aligned}\sum_{i,j=1}^{n} a_{lij}^{A_l}\cdot r_l^{A_lij}+\sum_{j=1}^{n} b_{lj}^{A_l}\cdot r_l^{A_lj}+c_{ll}^{A_l}\cdot r_l^{A_l}+\sum_{k=1,k\neq l}^{p} c_{lk}^{A_l}\cdot r_k^{A_l}&\\ \leqslant E^{A_l}(h)+D(h)+\eta\cdot r_l^{A_l}+\sum_{k=1,k\neq l}^{p} c_{lk}^{A_l}\cdot r_k^{A_l}&\,.\end{aligned} \tag{12.10}$$

But the assumption (12.5) and the definition of $g^{A_l}(h)$, cf. (12.4), yield

$$E^{A_l}(h)+D(h)+\eta r_l^{A_l}+\sum_{k=1,k\neq l}^{p} c_{lk}^{A_l}\cdot r_k^{A_l}<-\varepsilon(h)\,, \tag{12.11}$$

hence, (12.10) and (12.11) imply that

$$\sum_{i,j=1}^{n} a_{lij}^{A_l}\cdot r_l^{A_lij}+\sum_{j=1}^{n} b_{lj}^{A_l}\cdot r_l^{A_lj}+c_{ll}^{A_l}\cdot r_l^{A_l}+\sum_{k=1,k\neq 1}^{p} c_{lk}^{A_l}\cdot r_k^{A_l}<-\varepsilon(h)\,. \tag{12.12}$$

Since inequalities (12.12) and (12.1) are contradictory, we conclude that the maximum value $r_l^{A_l}$ satisfies the first formula of (12.3).

The proof of the second part of (12.3) can be obtained in a similar way.

This ends the proof of Lemma 4.

§ **13.** THEOREM 1. *Let us suppose that the functions* r_l^M $(l=1,\dots,p)$ *satisfy the assumptions of Lemma* 4; *cf.* § 12.

Under these assumptions 1° *the maximum values* $r_l^{A_l}$ $(l=1,\dots,p)$ *satisfy the linear system of algebraic inequalities*

$$r_l^{A_l}+\gamma\cdot\sum_{k=1,k\neq l}^{p} r_k^{A_k}\leqslant +g^{A_l}(h)\qquad (l=1,\dots,p)\,; \tag{13.1}$$

2° *The minimum values* $r_l^{B_l}$ $(l=1,\dots,p)$ *satisfy the linear system of algebraic inequalities*

$$r_l^{B_l}+\gamma\cdot\sum_{k=1,k\neq l}^{p} r_k^{B_k}\geqslant -g^{B_l}(h)\qquad (l=1,\dots,p)\,. \tag{13.2}$$

In the formula (13.1) (13.2) γ *denotes a negative number*:

$$\gamma = +\eta^{-1}\cdot\delta\ (\gamma<0)\,, \tag{13.3}$$

cf. Condition W_2 (§ 4); *and the functions* $g^{A_l}(h)$, $g^{B_l}(h)$ *are defined by* (12.4).