

Spectral Properties of the Certain Differential Operators of the Second and Fourth Order

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I. Introduction. The research which led to the present paper began with an observation that the spectral properties of an operator of the Sturm-Liouville type which were discussed in [1] may be proved for an arbitrary differential operator of the second order. We showed that this operator does not lower the number of sign changes. Applying the theorem of Levin and Stepanov ([5]) made it possible to prove some results in a simpler way than in [1] (and, besides without the assumption that the operator is self-adjoint). Then we noticed that the differential operator of the fourth order with constant coefficients can be considered as a composition of two operators of the second order.

Applying this fact we obtained the main result: a theorem about spectral properties of the operator of the fourth order.

II. Necessary definitions and theorems

Definition 1. Let f be a continuous function in $[a, b]$. We denote by $S(f)$ the number of sign changes of f .

For $f \equiv 0$ we put $S(f) := -1$

For $f \not\equiv 0$ we put $S(f) := s \Leftrightarrow \exists a = x_0 < x_1 < \dots < x_{s+1} = b \exists C \neq 0 f_{[x_i, x_{i+1}]} \neq 0$

and $C(-1)^i \cdot f_{[x_i, x_{i+1}]}$ has a constant sign for $i = 0, 1, 2, s$. When such the finite system $\{x_i\}$ does not exist we admit $S(f) = +\infty$

Definition 2. Let us consider an operator $A: C^a[a, b] \rightarrow C^m[a, b]$ D is a domain of A . A is said to be the operator which does not lower the number of sign changes on D when for every $u \in D$ we have the inequality $S(A[u]) \geq S(u)$.

From the definitions we deduce at once the following:

Remark 1. A composition of the finite number of operators which do not lower the number of sign changes is the operator which does not lower the number of sign changes.

Remark 2. Let A be the operator which does not lower the number of sign changes; f be a nontrivial continuous function in $[a, b]$. Let us define the operator \tilde{A} :

$$\tilde{A}[u] := f \cdot A[u] \quad \text{for } u \in D$$

If $S(f) = 0$, then A is the operator which does not lower the number of sign changes and $S(\tilde{A}[u]) = S(A[u])$.

We shall use the following definitions of the kind and order of the zero points of function f .

Definition 3. Let $f \in C[a, b]$. The interval $[c, d] \subset [a, b]$ is the zero point of function f .

$[c, d]$ is a node of $f \Leftrightarrow \exists \varepsilon_0 > 0 \forall \varepsilon > 0 [\varepsilon < \varepsilon_0 \Rightarrow f(c - \varepsilon)f(d + \varepsilon) < 0]$

$[c, d]$ is the adherent zero point of $f \Leftrightarrow \exists \varepsilon_0 > 0 \forall \varepsilon > 0 [[\varepsilon < \varepsilon_0 \Rightarrow f(c - \varepsilon)f(d + \varepsilon) > 0]$

Definition 4. Let f be of class C^k in a neighbourhood of x_0 . We say that $x_0 \in (a, b)$ is the zero point of f of the order k if

$$f(x_0) = f'(x_0) = \dots f^{(k-1)}(x_0) = 0 \quad \text{and } f^{(k)}(x_0) \neq 0$$

Let $J \subset [a, b]$; $\#J \geq k$. We get successive definition:

Definition 5: The system of k continuous functions $\{f_1, f_2, \dots, f_k\}$ defined in $[a, b]$ is said to be a system of Chebyshev in $[a, b]$ with respect to J when every nontrivial linear combination $\sum_{i=1}^k c_i f_i$ has at most $k-1$ zero points in J and changes the sign in $[a, b]$ at most $k-1$ times. Let f_1, f_2, \dots be a sequence of continuous functions in $[a, b]$. This sequence is said to be a Markov's series in $[a, b]$ with respect to J when for every $k \in \mathbb{N}$ $\{f_1, f_2, \dots, f_k\}$ is a system of Chebyshev.

We now give the main theorem. Consider the problem of eigenvalues and eigenfunctions for the equation:

$$(1) \quad L[u] = u^{(n)} + p_1 u^{(n-1)} + \dots + p_n u = \lambda \rho u$$

with boundary conditions

$$u^{(k_i)}(a) + \sum_{k < k_i} \gamma_{ik} u^{(k)}(a) = 0 \quad \text{for } i = 1, 2, \dots, m.$$

$$(2) \quad u^{(k_i)}(b) + \sum_{k < k_i} \gamma_{ik} u^{(k)}(b) = 0 \quad \text{for } i = m+1, \dots, n$$

$$\int_{[a, b]} u^2 d\mu = 1$$

at the end points of the interval $[a, b]$.

$$1 \leq m \leq n-1; \quad 0 \leq k_1 < \dots < k_m \leq n-1$$

$$0 \leq k_{m+1} < \dots < k_n \leq n-1$$

