

Spectral Properties of the Certain Differential Operators of the Second and Fourth Order

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I. Introduction. The research which led to the present paper began with an observation that the spectral properties of an operator of the Sturm-Liouville type which were discussed in [1] may be proved for an arbitrary differential operator of the second order. We showed that this operator does not lower the number of sign changes. Applying the theorem of Levin and Stepanov ([5]) made it possible to prove some results in a simpler way than in [1] (and, besides without the assumption that the operator is self-adjoint). Then we noticed that the differential operator of the fourth order with constant coefficients can be considered as a composition of two operators of the second order.

Applying this fact we obtained the main result: a theorem about spectral properties of the operator of the fourth order.

II. Necessary definitions and theorems

Definition 1. Let f be a continuous function in $[a, b]$. We denote by $S(f)$ the number of sign changes of f .

For $f \equiv 0$ we put $S(f) := -1$

For $f \not\equiv 0$ we put $S(f) := s \Leftrightarrow \exists a = x_0 < x_1 < \dots < x_{s+1} = b \exists C \neq 0 \int_{[x_i, x_{i+1}]} \neq 0$

and $C(-1)^i \cdot f_{[x_i, x_{i+1}]}$ has a constant sign for $i = 0, 1, 2, s$. When such the finite system $\{x_i\}$ does not exist we admit $S(f) = +\infty$

Definition 2. Let us consider an operator $A: C^a[a, b] \rightarrow C^m[a, b]$ D is a domain of A . A is said to be the operator which does not lower the number of sign changes on D when for every $u \in D$ we have the inequality $S(A[u]) \geq S(u)$.

From the definitions we deduce at once the following:

Remark 1. A composition of the finite number of operators which do not lower the number of sign changes is the operator which does not lower the number of sign changes.

Remark 2. Let A be the operator which does not lower the number of sign changes; f be a nontrivial continuous function in $[a, b]$. Let us define the operator \tilde{A} :

$$\tilde{A}[u] := f \cdot A[u] \quad \text{for } u \in D$$

If $S(f) = 0$, then A is the operator which does not lower the number of sign changes and $S(\tilde{A}[u]) = S(A[u])$.

We shall use the following definitions of the kind and order of the zero points of function f .

Definition 3. Let $f \in C[a, b]$. The interval $[c, d] \subset [a, b]$ is the zero point of function f .

$$[c, d] \text{ is a node of } f \Leftrightarrow \exists \varepsilon_0 > 0 \forall \varepsilon > 0 [\varepsilon < \varepsilon_0 \Rightarrow f(c-\varepsilon)f(d+\varepsilon) < 0]$$

$$[c, d] \text{ is the adherent zero point of } f \Leftrightarrow \exists \varepsilon_0 > 0 \forall \varepsilon > 0 [[\varepsilon < \varepsilon_0 \Rightarrow f(c-\varepsilon)f(d+\varepsilon) > 0]$$

Definition 4. Let f be of class C^k in a neighbourhood of x_0 . We say that $x_0 \in (a, b)$ is the zero point of f of the order k if

$$f(x_0) = f'(x_0) = \dots f^{(k-1)}(x_0) = 0 \quad \text{and } f^{(k)}(x_0) \neq 0$$

Let $J \subset [a, b]$; $\#J \geq k$. We get successive definition:

Definition 5: The system of k continuous functions $\{f_1, f_2, \dots, f_k\}$ defined in $[a, b]$ is said to be a system of Chebyshev in $[a, b]$ with respect to J when every nontrivial linear combination $\sum_{i=1}^k c_i f_i$ has at most $k-1$ zero points in J and changes the sign in $[a, b]$ at most $k-1$ times. Let f_1, f_2, \dots be a sequence of continuous functions in $[a, b]$. This sequence is said to be a Markov's series in $[a, b]$ with respect to J when for every $k \in \mathbb{N}$ $\{f_1, f_2, \dots, f_k\}$ is a system of Chebyshev.

We now give the main theorem. Consider the problem of eigenvalues and eigenfunctions for the equation:

$$(1) \quad L[u] = u^{(n)} + p_1 u^{(n-1)} + \dots + p_n u = \lambda \rho u$$

with boundary conditions

$$u^{(k_i)}(a) + \sum_{k < k_i} \gamma_{ik} u^{(k)}(a) = 0 \quad \text{for } i = 1, 2, \dots, m.$$

$$(2) \quad u^{(k_i)}(b) + \sum_{k < k_i} \gamma_{ik} u^{(k)}(b) = 0 \quad \text{for } i = m+1, \dots, n$$

$$\int_{[a, b]} u^2 d\mu = 1$$

at the end points of the interval $[a, b]$.

$$1 \leq m \leq n-1; \quad 0 \leq k_1 < \dots < k_m \leq n-1$$

$$0 \leq k_{m+1} < \dots < k_n \leq n-1$$

$-\gamma_{ik}$ are constant real numbers. We assume that q, p_i (for $i = 1, 2, \dots, n$) are continuous functions and q is positive in $[a, b]$. Let D be the set of all functions of class C^n in $[a, b]$ which satisfy the boundary conditions (2). So $L: D \rightarrow C[a, b]$.

Let us define:

$$I_1 := \begin{cases} [a, b] & \text{for } k_1 \neq 0 \text{ and } k_{m+1} \neq 0 \\ (a, b) & \text{for } k_1 = 0 \text{ and } k_{m+1} \neq 0 \\ [a, b) & \text{for } k_1 \neq 0 \text{ and } k_{m+1} = 0 \\ (a, b) & \text{for } k_1 = k_{m+1} = 0 \end{cases}$$

$$\sigma(x) := \int_a^x q(t) dt$$

We denote by I_σ the set of the points for which σ is increasing in $[a, b]$.

THEOREM of Levin and Stěpanov ([5]). *If the operator L does not lower the number of sign changes then problem (1) (2) has the following properties:*

1° *There exists a sequence of eigenvalues*

$$0 < |\lambda_1| < |\lambda_2| < \dots$$

Every λ_i is real and simple.

2° *The eigenfunction u_i corresponding to the eigenvalue λ_i has exactly $i-1$ zero points in (a, b) . All these zeros are simple. At the end points of the interval $[a, b]$ the order of zero points depends on the form of boundary conditions.*

3° *u_1, u_2, \dots is a Markov's series in $[a, b]$ with respect to $I_1 \cap I_\sigma$.*

Every nontrivial linear combination $c_k u_k + \dots + c_m u_m$ ($k \leq m$) has at least $k-1$ nodes and at most $m-1$ zero points in I_1 . This estimation holds true if we compute the adherent zero points doubly.

III. Spectral properties of the differential operator of the second order. Consider the problem of eigenfunctions for the equation:

$$(3) \quad \hat{L}[u] = -(\hat{p}u')' + \hat{q} \cdot u = \lambda \hat{q}u$$

with boundary conditions

$$(4) \quad \begin{aligned} \alpha_1 \cdot u(a) - \alpha_2 u'(a) &= 0 \\ \beta_1 u(b) + \beta_2 u'(b) &= 0 \\ \int_{[a, b]} u^2 d\mu &= 1 \end{aligned}$$

at the end points of the interval $[a, b]$, where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constant non-negative numbers fulfilling the conditions

$$\alpha_1^2 + \alpha_2^2 > 0 \quad \text{and} \quad \beta_1^2 + \beta_2^2 > 0.$$

We assume that $\hat{p}, \hat{q}, \hat{q}$ are continuous and positive functions in $[a, b]$. Let \hat{D} be the set of all functions of class C^1 in $[a, b]$ which satisfy the boundary conditions (4).

In [1] we find the following:

LEMMA 1. *If*

1° *f and $g = f' \cdot \hat{p}$ are of class C^1 in $[a, b]$;*

2° *f fulfills the boundary conditions (4) for $\alpha_2 = 0 \vee \beta_2 = 0$;*

3° *$f(x) \neq 0$ for $x \in (a, b)$.*

then there exists a point $\eta \in (a, b)$ such that $f(\eta) \cdot \hat{L}[f](\eta) > 0$.

Now we shall prove:

LEMMA 2. *The operator \hat{L} defined by (3) does not lower the number of sign changes on \hat{D} .*

Proof. If $f \equiv 0$ then $S(f) = -1 \leq S(L[f])$. Let $f \not\equiv 0$. It is obvious that $L[f] \not\equiv 0$ (because $\hat{q} > 0$). Denote $S(f) = s < \infty$. From the definition there exists a sequence $a = x_0 < \dots < x_{s+1} = b$ such that $f_{[x_i, x_{i+1}]} \not\equiv 0$ and $(-1)^i \cdot f_{[x_i, x_{i+1}]}$ has a constant sign (for $i = 1, \dots, s$).

Because f is continuous we get that $f(x_1) = f(x_2) = \dots = f(x_s) = 0$. Let $f_{[x_i, x_{i+1}]} \geq 0$. We can write that $f(x) > 0$ for $x \in (x_i, x_{i+1})$, when this inequality is not true we get a subinterval of (x_i, x_{i+1}) .

By Lemma 1 we know that in (x_i, x_{i+1}) there exists a point η_1 such that $\hat{L}[f](\eta_1) > 0$. In a similar way we can find $\eta_2 \in (x_{i+1}, x_{i+2})$ such that $\hat{L}[f](\eta_2) < 0$. So one sign change of f in $[x_i, x_{i+2}]$ implies at least one sign change of $L[f]$. The above reasoning transfers without alterations to the case $S(f) = \infty$. The proof is completed.

Consider the problem of eigenfunctions for the equation:

$$(5) \quad L[u] = -u'' + ru' + qu = \lambda \varrho \cdot u$$

with boundary conditions (4) where r, ϱ, q are continuous functions in $[a, b]$ and q and ϱ are positive in $[a, b]$. Let D be the set of all functions of class C^2 in $[a, b]$ which satisfy the boundary conditions (4).

LEMMA 3. *The operator L defined by (5) does not lower the number of sign changes on D .*

Proof. Let us take the operator

$$\hat{L}[u] = -(\hat{p}u')' + \hat{q}u \quad \text{for } u \in \hat{D},$$

where: $\hat{p}(x) = e^{F(x)} > 0$ for $x \in [a, b]$

$$\hat{q}(x) = q(x) \cdot \hat{p}(x) > 0$$

$$F'(x) = -r(x)$$

So $\hat{L}[u](x) = -[e^{F(x)}u'(x)]' + q(x)e^{F(x)}u(x)$.

According to Lemma 2 it follows that \hat{L} does not lower the number of sign changes on \hat{D} . But we notice that

$$L[u] = \frac{1}{\hat{p}} \hat{L}[u] \quad \text{for } u \in D$$

In fact:

$$\frac{\hat{L}[u](x)}{\hat{p}(x)} = \frac{-1}{e^{F(x)}} [F'(x)e^{F(x)}u'(x) + e^{F(x)}u''(x)] + q(x) \cdot u(x)$$

$$\frac{\hat{L}u}{\hat{p}(x)} = -u''(x) + r(x)u'(x) + q(x)u(x).$$

By the definition $S\left(\frac{1}{\hat{p}}\right) = 0$. Applying Remark 1 we have our thesis.

Now we can formulate the theorem about spectral properties of the differential operator of the second order.

THEOREM 1. *Problem (5) (4) has the following properties:*

1° *There exists a sequence of eigenvalues*

$$0 < \lambda_1 < \lambda_2 < \dots; \lim_{n \rightarrow \infty} \lambda_n = \infty$$

Every λ_i is real and simple.

2° *The eigenfunction u_i corresponding to the eigenvalue λ_i has exactly $i-1$ zero points in (a, b) . All these points are simple. At the end points of the interval $[a, b]$ the order of zero points depends on the form of boundary conditions (if $\alpha_2 \neq 0$ then $u_i(a) \neq 0$, for $\alpha_2 = 0$ u_i has the simple zero point in a ; if $\beta_2 \neq 0$ then $u_i(b) \neq 0$, for $\beta_2 = 0$ u_i has the simple zero points in b).*

3° *u_1, u_2, \dots is a Markov's series in $[a, b]$ with respect to $I_1 \cap I_\sigma$.*

Every nontrivial linear combination $c_k u_k + \dots + c_m u_m$ ($k \leq m$) has at least $k-1$ nodes and at most $m-1$ zero points in α_1 . This estimation holds true if we compute the adherent zero points doubly.

Proof. This theorem follows at once from Theorem of Levin and Stěpanov and Lemma 3. We have only one difference between 1° and the thesis of Levin and Stěpanov. But q, ϱ are positive, so we can define eigenvalues of \hat{L} by variation methods (see [3]). Then we have that $\lambda_n > 0$ (for $n \in N$) and $\lim_{n \rightarrow \infty} \lambda_n = \infty$.

Corroboration. If L is a linear operator, ϱ is a positive function and $\{\lambda_n\}, \{u_n\}$ mean the sequences of eigenvalues and eigenfunctions for the equation $L[u] = \lambda \varrho \cdot u$ with conditions (4) then $\{\lambda_n + c\}, \{u_n\}$ are the sequences of eigenvalues and eigenfunctions for the equation $(L + I c \varrho)[u] = (\lambda + c) \varrho u$ with conditions (4) where c is an arbitrary real constant.

From the above corroboration we have the following:

CONCLUSION. *In the equation (5) it is not necessary to assume that $q > 0$. Without this assumption problem (5) (4) has every spectral properties mentioned in Theorem 1 but condition 1° takes the form*

1^o *There exists a sequence of eigenvalues*

$$\lambda_1 < \lambda_2 < \dots \quad \text{and} \quad \lim_{n \rightarrow \infty} \lambda_n = \infty.$$

Proof. Since q, ϱ are continuous functions in $[a, b]$ and ϱ is positive, there exists a number $c \in \mathbb{R}$ such that $q(x) + c\varrho(x) > 0$ for $x \in [a, b]$. Put $q_1(x) = q(x) + c\varrho(x)$.

Then problem (5) (4) is equivalent to (5') (4), where

$$(5') \quad -u'' + ru' + q_1u = (\lambda + c)\varrho u.$$

Since r, q_1 satisfy the assumptions of Theorem 1, we have our conclusion.

IV. Spectral properties of the differential operator of the fourth order with constant coefficients

Let

$$(6) \quad \tilde{L}_4[u] = (L_2 \circ L_1)[u] = \lambda \varrho u,$$

where $L_i[u] = -u'' + q_i u$ for $i = 1, 2$.

We assume that ϱ, q_i ($i = 1, 2$) are continuous and positive functions in $[a, b]$. We shall consider two kinds of boundary conditions:

$$(7) \quad \begin{aligned} u(a) = u(b) = u'(a) = u'(b) = 0 \\ \int_{[a, b]} u^2 d\mu = 1 \end{aligned}$$

or

$$(8) \quad \begin{aligned} u(a) = u(b) = u''(a) = u''(b) = 0. \\ \int_{[a, b]} u^2 d\mu = 1 \end{aligned}$$

Let us denote by D_1 the set of all functions of class C^4 in $[a, b]$ which satisfy the boundary conditions (7). Analogously D_2 is the set of functions of class C^4 in $[a, b]$ which satisfy the boundary conditions (8).

LEMMA 4. *The operator \tilde{L}_4 defined by (6) does not lower the number of sign changes on D_2 .*

Proof. Consider the following problem:

$$v = L_1[u] = -u'' + q_1u \quad \text{for } u \in D_2$$

with conditions $u(a) = u(b) = 0$. According to Lemma 2 we have the inequality $S(v) \geq S(u)$. But $v(a) = v(b) = 0$. Applying the above Lemma for the second time we get:

$$S(\tilde{L}_4[u]) = S(L_2[v]) \geq S(v) \geq S(u),$$

which was to be shown.

We shall prove the similar Lemma in the space D_1 . For this purpose we need the following

LEMMA 5 (see [6]). *If f is of class C^N ($2 \leq N < \infty$) in a neighbourhood of $x_0 \in [a, b]$ and if x_0 is the zero point of f of the order N then*

$$\lim_{x \rightarrow x_0} \frac{(x-x_0)^2 L_1[f](x)}{f(x)} = -N(N-1)$$

LEMMA 6. *The operator \tilde{L}_4 defined by (6) does not lower the number of sign changes on D_1 .*

Proof. Let us take $u \in D_1$. We denote $L_1[u] = v$. All assumptions of Lemma 2 hold. Hence we have $S(v) \geq S(u)$. Let $S(u) = s$. By the definition there exists a sequence $a = x_0 < x_1 \dots < x_{s+1} = b$ such that $u_{[x_i, x_{i+1}]}$ is nontrivial and $(-1)^i u_{[x_i, x_{i+1}]}$ has a constant sign (for $i = 0, 1, \dots, s$). Let us restrict to the interval $[a, x_1]$. Let $u_{[a, x_1]} \geq 0$.

Function u satisfies the assumptions of Lemma 1 and so there exists $\eta_0 \in (a, x_1)$ such that $u(\eta_0)v(\eta_0) > 0$. Since a is the zero point of u of the second order we have by Lemma 5 that $u(\eta)v(\eta) < 0$ in a certain right — side neighbourhood of a . Hence there exists $a_1 \in (a, x_1)$ such that $v(a_1) = 0$.

In a similar way we are able to show that there exists $b_1 \in (x_s, b)$ such that $v(b_1) = 0$. Let us consider the operator $L_2[v] = \tilde{L}_4[u]$ with conditions $v(a_1) = v(b_1) = 0$. Hence by Lemma 2 have:

$$S(\tilde{L}_4[u]) = S_{[a, b]}(L_2[v]) \geq S_{[a_1, b_1]}(L_2[v]) \geq S_{[a_1, b_1]}(v) \geq S_{[a_1, b_1]}(u) = S_{[a, b]}(u).$$

This completes the proof.

Corroboration. Lemma 6 holds true for the operator

$$\tilde{L}_4 = L_2 \circ L_1, \quad \text{where } L_i u = -u'' + r_i u' + q_i u \quad (i = 1, 2).$$

The above proof is applicable without any changes.

Consider the problem of eigenfunctions for the equation

$$(9) \quad L_4 u = u^{(4)} - B u'' + C u = \lambda \varrho u$$

with the boundary conditions (7) or (8). We assume that B, C are real and positive numbers which satisfy the inequality $B^2 \geq 4C$. Function ϱ is continuous and positive in $[a, b]$.

Now we shall prove the main theorem on spectral properties for the operator (9).

THEOREM 2. *Problem (9) (7) [or (9) (8)] has the following spectral properties:*

1° *There exists a sequence of eigenvalues*

$$0 < |\lambda_1| < |\lambda_2| < \dots$$

Every λ_i is real and simple.

2°. The eigenfunction u_i corresponding to the eigenvalue λ_i has exactly $i-1$ zero points in (a, b) . All these zeros are simple.

3° u_1, u_2, \dots is a Markov's series in $[a, b]$ with respect to $I_1 \cap I_\sigma$. Every nontrivial linear combination $\sum_{i=k}^m c_i u_i$ ($k \leq m$) has at least $k-1$ nodes and at most $m-1$ zero points in I_1 . This estimation holds true if we compute the adherent zero points doubly.

Proof. Let us notice that:

$$L_4[u] = -\left(-u'' + \frac{B - \sqrt{B^2 - 4C}}{2}u\right)'' + \frac{B + \sqrt{B^2 - 4C}}{2}\left(-u'' + \frac{B - \sqrt{B^2 - 4C}}{2}u\right).$$

So $L_4[u] = L_2 \circ L_1[u]$, where $L_i[u] = -u'' + q_i u$ ($i = 1, 2$) and

$$q_1 = \frac{B - \sqrt{B^2 - 4C}}{2} > 0$$

$$q_2 = \frac{B + \sqrt{B^2 - 4C}}{2} > 0.$$

According to Lemma 6 (or Lemma 4 in the second version) we have at once our thesis.

Remark 3. Theorem 2 holds true for the operator $L_4[u] = u^{(4)} - Bu'' + Cu$, where B, C are continuous and positive functions which satisfy the inequality

$$[B(x)]^2 \geq 4C(x) \quad \text{for } x \in [a, b] \quad \text{if } B - \sqrt{B^2 - 4C}$$

or $B + \sqrt{B^2 - 4C}$ is a constant number. The proof is trivial.

Remark 4. Theorem 2 holds true for the operator $L_4[u] = f \cdot L_4[u]$ if f is a nontrivial continuous function and f does not change the sign in $[a, b]$.

References

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