

On Invariant Measures Supported on the Compact Sets

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0. Introduction. The problem of existence of an invariant measure has been often considered since Prodi [6], Foiaş [3] and Hopf [5] stated that the problem an invariant measure corresponds to the problem of turbulence.

This paper is a continuation of the papers [1], [2] in which the existence of invariant measure has been proved for the dynamical system given by some partial differential equation. As far as the applications in biology [4] are concerned, it is not important for the invariant measure to be supported on the whole space of continuous functions, but it is very important for it to be supported on the whole set where this system has the biological sense. This problem is being solved in this paper.

1. The existence of an invariant measure

In this section we consider the equation

$$\frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = \gamma u$$

in the set

$$0 \leq x \leq 1 \quad t \geq 0$$

with the initial condition

$$u(0, x) = v(x) \quad 0 \leq x \leq 1.$$

This problem generates a semi-dynamical system according to the formula

$$(T_t v)(x) = u(t, x) \quad 0 \leq x \leq 1.$$

This system may be considered on the space V_α of all Lipschitz functions vanishing in 0 and satisfying the condition

$$|v'(x)| \leq \alpha x^{\gamma-1}$$

THEOREM 1. *If $\gamma > 1$, then there exists a non-negative measure μ_α defined on Borel subsets of V_α and satisfying the conditions:*

(i) μ_α is T_t -invariant i.e. for all $t > 0$ and for a Borel subset E of V_α , $\mu_\alpha(T_t^{-1}(E)) = \mu_\alpha(E)$.

(ii) $\mu_\alpha(U) > 0$ for every open non empty subset U of V_α .

(iii) μ_α is probabilistic i.e. $\mu_\alpha(V_\alpha) = 1$,

(iv) μ_α is ergodic i.e. for every T_t -invariant set E

$$\mu_\alpha(E) = \mu_\alpha(E)^2,$$

(v) $\mu_\alpha(E_0) = 0$, where E_0 is the set of all the periodic points.

We shall show more, namely that $\mu_\alpha(E'_0) = 0$ when $E'_0 = \{v | \exists s: T_s v \in E_0\}$. We denote the set of all non-negative functions from V_α , by V_α^+ .

THEOREM 2. *Theorem 1 is true also if we replace the space V_α by V_α^+ .*

To prove these theorems we shall use some lemmas.

2. Auxiliary lemmas

LEMMA 1. *Let $\{p_i\}$ be a sequence satisfying the conditions*

(a) $p_0 = 0$

(b) $\forall i > 0 \quad p_i > 0$

(c) $\sum_{i=1}^{\infty} p_i = 1$.

Let's define the function $q^*: [0, 1] \rightarrow [0, 1]$ by the formula

$$\begin{cases} q^*(\xi) = p^{-1}(\xi - \sum_{i=1}^n p_i) & \text{for } \xi \in (\sum_{i=1}^{n-1} p_i, \sum_{i=1}^n p_i] \\ q^*(0) = 0. \end{cases}$$

Thus the Lebesgue measure is ergodic with respect to q .

Let's consider now a semi-dynamical system $T_t: X \rightarrow X$ on a σ -compact metrizable space X .

Definition. A system $\{T_t\}$ is called *surjective* if for every $t > 0$, T_t is a surjection.

Let a non-negative Borel measure on X be given, such that for each Borel set E

$$\mu(T_t^{-1}(E)) = E.$$

Let's consider the function: $f = f_E: [0, 1] \rightarrow R$, given by the formula

$$f_E(t) = \mu(T_t^{-1}(E)).$$

LEMMA 2. f_E is measurable.

By $\text{Lip } [0, 1]$ we denote the set of all Lipschitz functions on $[0, 1]$.

LEMMA 3. *Let n_0 be a positive integer. There exists such a family \mathcal{P} of polynomials with rational coefficients that*

a) $\overline{\mathcal{P}} = \text{Lip } [0, 1]$

