

## Some Properties of Generalized Systems

by Janina KLAPYTA

The present paper deals with some properties of generalized pseudo-dynamical systems  $(X, G, \lambda)$  in the sense of [4] (see Def. 1 below) for which we assume additionally (A)  $(X, v)$  is a topological space with topology  $v$  and  $(G, +; <)$  is an abelian, ordered semigroup with the neutral element 0, which is also the minimal element.

Starting from the notion of recursivity according to Bhatia and Szegö [1], we transfer the definitions of Poisson stability, completely unstability, dispersivity and parallelizability of classical dynamical systems to generalized systems satisfying the above assumptions. The relations between these notions are considered. We also point out some differences between the notions defined for the "single-valued" systems and the "multi-valued" ones.

I would like to thank Professor A. Pelczar for his valuable advice.

### Notations and definitions

Notations and definitions admitted in this paper are just the same as in [3] and most of them are those used in [2], [4] and [5].

We denote by  $(\mathcal{A}(X), 2^v)$  the space of all non empty subsets of a topological space  $(X, v)$  with the topology  $2^v$  generated by families of the form

$$\langle V_1, \dots, V_n \rangle := \{B \in \mathcal{A}(X) : B \subset \bigcup_{i=1}^n V_i, B \cap V_i \neq \emptyset \text{ for } i = 1, \dots, n\}$$

corresponding to all finite systems of subsets of  $X$ , open in topology  $v$ . Here " := " denotes the equality by definition. By  $\mathcal{C}(X)$  we denote the family of all non empty and compact subsets of  $X$ .

For an arbitrary  $A \in \mathcal{A}(X)$  we denote by  $\mathcal{U}_A, \mathcal{V}_A, \dots$  neighbourhoods of the set  $A$  interpreting as a point of the space  $(\mathcal{A}(X), 2^v)$ , i.e. the family  $\mathcal{V}_A \in 2^v$ ,  $A \in \mathcal{V}_A$ . We denote by  $U_A, V_A, \dots$  the open sets in the space  $(X, v)$  which contain the set  $A$ , i.e.  $V_A \in v$ ,  $A \subset V_A$ .

The closure of the set  $A \subset X$  in the space  $(X, v)$  is denoted by  $\bar{A}$ , while the closure of the family  $\mathcal{B} \subset \mathcal{A}(X)$  interpreting as a subset of the space  $(\mathcal{A}(X), 2^v)$  is denoted by  $\text{cl } \mathcal{B}$ .

In order to make the notations shorter we introduce the map

$$\sigma: \mathcal{A}(\mathcal{A}(X)) \ni \mathcal{B} \rightarrow \bigcup \{B: B \in \mathcal{B}\} \in \mathcal{A}(X).$$

The generalized pseudo-dynamical system generalizing the notion of the pseudo-dynamical system has been defined in [4]:

Definition 1. A triplet  $(X, G, \lambda)$  is called a *generalized pseudo-dynamical system*, when  $X$  is an abstract space,  $(G, +)$  is an abelian semigroup with the neutral element  $0$  and  $\lambda$  is a map from  $G \times X$  into  $\mathcal{A}(X)$ , such that for every  $t, s \in G$ ,  $x \in X$ ,  $A \in \mathcal{A}(X)$  the conditions

$$(1) \quad \lambda_0(x) = \{x\},$$

$$(2) \quad \lambda_t(\lambda_s(A)) \subset \lambda_{t+s}(A) = \lambda_{s+t}(A)$$

are satisfied, where  $\lambda_t(x) := \lambda(t, x)$  and  $\lambda_t(A) := \sigma\{\lambda_t(x): x \in A\}$ .

A generalized pseudo-dynamical system  $(X, G, \lambda)$  is called *regular*, when the additional condition

$$(2') \quad \lambda_t(\lambda_s(A)) = \lambda_{t+s}(A)$$

is satisfied.

In the generalized pseudo-dynamical system  $(X, G, \lambda)$  we admit the following definitions (see [5]).

Definition 2. The set

$$(3) \quad \lambda(A) := \sigma\{\lambda_t(A): t \in G\}$$

is called *the zone of emission of the set*  $A \in \mathcal{A}(X)$  in the generalized system  $(X, G, \lambda)$ . We put  $\lambda(z) := \lambda(\{z\})$  for every  $z \in X$ .

Definition 3. The family of sets

$$(4) \quad T_\lambda(A) := \{\lambda_t(A): t \in G\}$$

is called *the trajectory of the set*  $A \in \mathcal{A}(X)$  in the generalized system  $(X, G, \lambda)$ .

Definition 4. A family  $\mathcal{B} \subset \mathcal{A}(X)$  is called *invariant* in  $(X, G, \lambda)$  if  $T_\lambda(B) \subset \mathcal{B}$  for every  $B \in \mathcal{B}$ .

A set  $Z \subset X$  is called *invariant* in  $(X, G, \lambda)$  if  $\lambda(z) \subset Z$  for every  $z \in Z$ .

Let  $A$  be a set belonging to  $\mathcal{A}(X)$ . Under the assumptions we gave at the beginning of this paper and relative to the generalized pseudo-dynamical systems  $(X, G, \lambda)$  we admit definitions of  $\prec$ -limit set and  $\prec$ -prolongational limit set of the trajectory of the set  $A$ .

Definition 5. *The  $\prec$ -limit set of the trajectory of the set*  $A \in \mathcal{A}(X)$  is the set

$$(5) \quad \Omega(A) := \bigcap_{t \in G} (\text{cl}\{\lambda_s(A): t \prec s, s \in G\}).$$

Definition 6. The  $\prec$ -prolongational limit set of the trajectory is the family

$$(6) \quad J^\Omega(A) := \{B \in \mathcal{A}(X) : \forall \mathcal{V}_B \forall \mathcal{V}_A \forall t \in G \exists s \in G, t \prec s, \exists C \in \mathcal{V}_A \text{ such that } \lambda_s(C) \in \mathcal{V}_B\}.$$

Other variants of  $\prec$ -limit sets and  $\prec$ -prolongational limit sets in generalized systems together with their properties and connections between them are described and discussed in [3], [5].

In every "single-valued" semi-system  $(X, G, \pi)$  in the sense of [4] we shall denote by  $\Lambda(x)$  the limit set of the point  $x \in X$  and by  $J(x)$  the prolongational limit set of the point  $x \in X$ .

We admit that in the examples which we shall consider in this paper the space  $R^2$  is provided with the natural topology; so  $(R^2, v)$  is with the topology  $v$  generated by

$$(7) \quad K(x, r) := \{y \in R^2 : (x_1 - y_1)^2 + (x_2 - y_2)^2 < r^2\}$$

for  $x \in R^2, r \in R_+$ .

### Poisson unstability, completely unstability and dispersiveness of generalized dynamical systems

Let  $(X, G, \lambda)$  be a given generalized pseudo-dynamical system in the sense of Definition 1, satisfying the assumptions (A). In this system some notions are defined, generalizing the corresponding notions in semi-dynamical "single-valued" systems (see [1], [4]).

Definition 7. A family  $\mathcal{B} \subset \mathcal{A}(X)$  will be said to be *recursive* with respect to a family  $\mathcal{C} \subset \mathcal{A}(X)$  if

$$(8) \quad \forall t \in G \exists s \in G, t \prec s, \forall C \in \mathcal{C} : \lambda_s(C) \in \mathcal{B}.$$

Let  $A$  be a set belonging to  $\mathcal{A}(X)$ .

Definition 8. The trajectory of the set  $A$  is *Poisson stable* ( $T_\lambda(A)$  is *P-stable*) if and only if every neighbourhood of the set  $A$  in the space  $(\mathcal{A}(X), 2^v)$  is a recursive family with respect to the family  $\{A\}$ , i.e.

$$(9) \quad \forall \mathcal{V}_A \forall t \in G \exists s \in G, t \prec s : \lambda_s(A) \in \mathcal{V}_A.$$

It is easy to see, that for this system we have the following characterization of Poisson stability.

**THEOREM 1.** *The trajectory of the set  $A$  is Poisson stable if and only if  $A \in \Omega(A)$ .*

Remark 1. This theorem gives the compatibility of Definition 8 of Poisson stability of the trajectory of the set  $A$  with the definition of this notion introduced in [5].

Definition 9. A set  $A \in \mathcal{A}(X)$  is called *non-wandering* if every neighbourhood of  $A$  in the space  $(\mathcal{A}(X), 2^v)$  is self recursive, i.e.

$$(10) \quad \forall \mathcal{V}_A \forall t \in G \exists s \in G, t < s, \exists C \in \mathcal{V}_A: \lambda_s(C) \in \mathcal{V}_A.$$

Here we give also a characterization of non-wandering sets.

THEOREM 2. A set  $A \in \mathcal{A}(X)$  is non-wandering if and only if  $A \in J^\Omega(A)$ .  
We have the following:

THEOREM 3. Every set belonging to a closure of the family of sets having a Poisson stable trajectory is non-wandering.

Proof. Let  $\mathcal{P} \subset \mathcal{A}(X)$  be the family of sets having a Poisson stable trajectory. We take an arbitrary set  $B$  which belongs to  $\text{cl } \mathcal{P}$ . For every neighbourhood  $\mathcal{V}_B$  there is a set  $P \in \mathcal{P} \cap \mathcal{V}_B$ . Take  $\mathcal{V}_P := \mathcal{V}_B$  and observe that the thesis of this theorem follows from Poisson stability of the trajectory of set  $P$ .

Let  $A$  be a set belonging to  $\mathcal{A}(X)$ .

Definition 10. The trajectory of the set  $A$  is *Poisson unstable* if it is not Poisson stable.

Definition 11. A set  $A$  is *wandering* if it is not non-wandering.

Definition 12. The generalized pseudo-dynamical system  $(X, G, \lambda)$  is called

- (11) *Poisson unstable* if the trajectory of every set  $A \in \mathcal{A}(X)$ ,  $A \neq X$  is Poisson unstable,
- (12) *completely unstable* if every set  $A \in \mathcal{A}(X)$ ,  $A \neq X$  is wandering
- (13) *dispersive* if for every pair of sets  $A, B \in \mathcal{A}(X)$ ,  $A \neq X$ ,  $B \neq X$  there exist neighbourhoods  $\mathcal{V}_A, \mathcal{V}_B \in 2^v$  such that the family  $\mathcal{V}_B$  is not recursive with respect to the family  $\mathcal{V}_A$ . Directly by definition we have some characterizations of dispersivity.

THEOREM 4. The following conditions are equivalent:

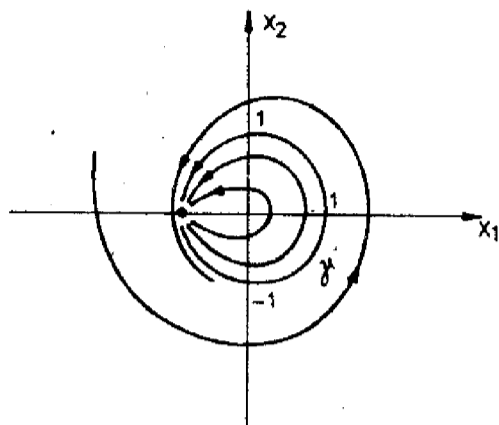
- (i) the system  $(X, G, \lambda)$  is dispersive
- (ii) for every pair of sets  $A, B \in \mathcal{A}(X)$ , where  $A \neq X$ ,  $B \neq X$  we have  $B \notin J^\Omega(A)$
- (iii) for every set  $A \in \mathcal{A}(X)$ , where  $A \neq X$ , we have  $J^\Omega(A) = \emptyset$ .

Here is an immediate conclusion from the definitions and theorems.

Remark 2. Analogously as in "single-valued" systems one obtains the following implications

$$(11) \Rightarrow (12) \Rightarrow (13).$$

Example 1. Let  $(X, \nu)$  be a space  $X = \mathbb{R}^2 \setminus \{(-1, 0)\}$  with the topology induced from  $(\mathbb{R}^2, \nu)$  (see (7)). There is given a semi-system  $(X, R_*, \pi)$  (see [1]), which trajectories are geometrically described:



Let us consider the generalized system  $(X, R_*, \lambda)$  with

$$(14) \quad \lambda_t(x) := \{\pi(t, x)\} \quad \text{for } t \in R_*, x \in X.$$

It is easily seen that the semi-system  $(X, R_*, \pi)$  is Poisson unstable, whilst the generalized system  $(X, R_*, \lambda)$  is not Poisson unstable, because for example  $\gamma \in \Omega(\gamma)$ , with

$$\gamma := S((0, 0), 1) \setminus \{(-1, 0)\}, S((0, 0), 1) := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}.$$

The following example is pointing out the main reasons for differences between the notions defined for the “single-valued” systems and the “multi-valued” ones considered in this chapter.

Example 2. Let  $(\mathbb{R}^2, R, \pi)$  be a dynamical system generated in the usual way by the solutions of  $y' = 0, x' = 1$ . I.e.

$$\pi(t, (\hat{x}, \hat{y})) := (x(t, (\hat{x}, \hat{y})), y(t, (\hat{x}, \hat{y}))), \quad \text{where } (x(\cdot, (\hat{x}, \hat{y})), y(\cdot, (\hat{x}, \hat{y})))$$

is the (unique) solution of the Cauchy initial value problem. We take the generalized system  $(\mathbb{R}^2, R_*, \lambda)$  which is defined by  $(\mathbb{R}^2, R, \pi)$  in such a way as in Example 1 (see (14)). It is evident that the system  $(\mathbb{R}^2, R, \pi)$  is parallelizable and the semi-system  $(\mathbb{R}^2, R_*, \pi)$  is dispersive (see [4]), however the generalized system  $(\mathbb{R}^2, R_*, \lambda)$  is not even Poisson unstable. It is so, because we have  $\Omega(A_0) = \{A_0\}$  for  $A_0 = \pi(P_0)$ , though  $\Omega(A_1) = \emptyset$  for  $A_1 = \pi_+(P_0), P_0 \in \mathbb{R}^2$ .

Differences between these properties of the dynamical semi-system  $(\mathbb{R}^2, R_*, \pi)$  and the generalized dynamical system  $(\mathbb{R}^2, R_*, \lambda)$  are caused by the possibility of taking into consideration not only points but also sets and by the fact that the semi-system  $(\mathbb{R}^2, R_*, \pi)$  is derived from the system  $(\mathbb{R}^2, R, \pi)$  and practically gives us the possibility of going back without limit, i.e. the possibility of continuous inflowing of the points on the place of shifting ones. It causes that for the trajectory of a point  $P_0$  in the “single-valued” semi-system, treated as a set (see  $A_0$ ) in the generalized system, we have a family of  $\leftarrow$ -limit sets equal to the family which contains the very trajectory as a single element. It is so, however the limit sets of points are always empty in the “single-valued” semi-system. If we cut

off the trajectory (see  $A_1$ ), then the family of  $\prec$ -limit sets is empty in the generalized system, too.

Example 3. Let us take the generalized system  $(Y, R_*, \lambda)$  derived from the system considered in example 1 by changing the space, where  $(Y, v)$  is a space

$$Y := R^2(B((0, 0), 1) \cup \{(-1, 0)\}), \quad B((0, 0), 1) := \{x \in R^2: x_1^2 + x_2^2 \leq 1, x_2 < 0\}$$

with the topology induced by  $(R^2, v)$  (see (7)). The generalized system  $(Y, R_*, \lambda)$  is Poisson unstable, but is not completely unstable, since for every point  $x_0 \in \gamma \cap Y$  (see ex. 1) we have

$$J^\Omega(x_0) = \{\{x\}: x \in \gamma \cap Y\}.$$

Example 3 points out the differences between the notion of Poisson unstability and the notion of complete unstability of the generalized system.

### Parallelizable dynamical systems

Definition 13. A generalized pseudo-dynamical system  $(X, G, \lambda)$  (in the sense of Definition 1), where  $(X, v)$  is a topological space is *the generalized dynamical system* if the map is continuous.

In order to state some theorems on connections between properties of the generalized dynamical systems (see Theorem 8) we admit the following natural generalization of the notion of parallelizability of the system.

Definition 14. A generalized dynamical system  $(X, G, \lambda)$  is called *pseudo-parallelizable* if there are the families  $\chi \subset \mathcal{A}(X)$ ,  $\mathcal{S} \not\subseteq \chi$  and a homeomorphism  $h: \chi \rightarrow G \times \mathcal{S}$  such that

$$(15) \quad h(\chi(t, A)) = (t, A)$$

$$(16) \quad \chi = \bigcup \{\lambda_t(\mathcal{S}): t \in G\}.$$

A generalized dynamical system is *parallelizable* if it is pseudo-parallelizable with  $\chi = \mathcal{A}(X)$ .

Definition 15. A family  $\mathcal{S} \not\subseteq \mathcal{A}(X)$  is called *a section of a generalized dynamical system*  $(X, G, \lambda)$  if for every  $A \in \mathcal{A}(X)$  there are unique  $\tau(A) \in G$  and  $B_A \in \mathcal{S}$  such that  $\lambda_{\tau(A)}(B_A) = A$ .

THEOREM 5. If  $\mathcal{S}$  is a section of the generalized dynamical system  $(X, G, \lambda)$  with the map  $\tau(\cdot)$  continuous on  $\mathcal{A}(X)$ , then the family  $\mathcal{S}$  is closed in  $(\mathcal{A}(X), 2^v)$ .

The proof is evident if we observe, that for every set  $B \in \text{cl}\mathcal{S}$  we have  $\tau(B) = 0$  from the continuity of the map  $\tau(\cdot)$ .

The following necessary condition of parallelizability of the generalized dynamical system is an immediate conclusion from the Definitions 14 and 15.

**THEOREM 6.** *If the generalized dynamical system  $(X, G, \lambda)$  is parallelizable, then there is a family  $\mathcal{S}$  which is section of  $(X, G, \lambda)$  with the map  $\tau(\cdot)$  continuous on  $\mathcal{A}(X)$ .*

Assuming some additional conditions about  $\mathcal{A}(X)$  we can prove the following

**THEOREM 7.** *If  $(\mathcal{A}(X), 2^v)$  is a compact Hausdorff space and  $(X, G, \lambda)$  is a generalized dynamical system, then parallelizability of the system  $(X, G, \lambda)$  follows from the existence of the section with the continuous map  $\tau(\cdot)$ .*

**Proof.** We define the map  $h: \mathcal{A}(X) \rightarrow G \times \mathcal{S}$  in the following way  $h(A) := (\tau(A), B_A)$  (see definition 15). Since  $h^{-1}(\tau, B) = \lambda_\tau(B)$ , we have the continuity of the map  $h^{-1}$ . We shall prove that the map  $h$  is continuous. Suppose there is a set  $A \in \mathcal{A}(X)$  in which  $h$  is not continuous, i.e. there is a neighbourhood  $\mathcal{V}_{(\tau(A), B_A)}^0$  such that for every neighbourhood  $\mathcal{V}_A$  there is a set  $C \in \mathcal{V}_A$  such that  $(\tau(C), B_C) \notin \mathcal{V}_{(\tau(A), B_A)}^0$ . Let  $\mathcal{V}_{(\tau(A), B_A)}^0 = V_\tau^0(A) \times \mathcal{V}_{B_A}^0$ . We take an arbitrary  $\tilde{V}_{\tau(A)} \subset V_\tau^0(A)$ . From the continuity of the map  $\tau(\cdot)$  we have

$$\forall \tilde{V}_{\tau(A)} \exists \tilde{\mathcal{V}}_A: \forall C \in \tilde{\mathcal{V}}_A \quad \tau(C) \in \tilde{V}_{\tau(A)}.$$

For every  $\tilde{V}_{\tau(A)} \subset V_\tau^0(A)$  and for every  $\mathcal{V}_A \subset \tilde{\mathcal{V}}_A$  there is  $C \in \mathcal{V}_A$  such that  $\tau(C) \in \tilde{V}_{\tau(A)}$  and  $B_C \notin \mathcal{V}_{B_A}^0$ . From the compactness of  $\mathcal{A}(X)$  there is  $B \in \mathcal{A}(X)$  such that

$$\forall \mathcal{V}_B \exists C \in \mathcal{V}_A: B_C \in \mathcal{V}_B, B \neq B_A.$$

There exist separated neighbourhoods  $\mathcal{V}_A$  and  $\mathcal{V}_{A^0}$  because  $A = \lambda_{\tau(A)}(B_A) \neq \lambda_{\tau(A)}(B) =: A^0$ . In virtue of the continuity of the map  $\lambda$  we take  $V_{\tau(A)} \times \mathcal{V}_B$  for the neighbourhood  $\mathcal{V}_{A^0}$ . There is  $C \in \mathcal{V}_A$  such that  $\tau(C) \in V_{\tau(A)}$  and  $B_C \in \mathcal{V}_B$ . It is clear that  $\lambda_{\tau(C)}(B_C) = C \in \mathcal{V}_{A^0}$  and this gives the contradiction, since the neighbourhoods  $\mathcal{V}_A$  and  $\mathcal{V}_{A^0}$  are separated.

Analogously to "single-valued" dynamical systems obtains:

**THEOREM 8.** *If the generalized dynamical system  $(X, G, \lambda)$  is parallelizable, then it is dispersive.*

**Proof.** Suppose there are sets  $A, B \in \mathcal{A}(X)$  such that  $A \neq X$ ,  $B \neq X$  and  $B \in J^\Omega(A)$ . We take arbitrary bounded\* neighbourhoods  $V_{\tau(A)}$ ,  $V_{\tau(B)}$  and choose for them by continuity of  $\tau(\cdot)$   $\mathcal{V}_A$  and  $\mathcal{V}_B$ , accordingly. Since  $B \in J^\Omega(A)$ , we have for chosen  $\mathcal{V}_A, \mathcal{V}_B$  which implies

$$\forall t \in G \exists s \in G, t < s, \exists C \in \mathcal{V}_A: \lambda_s(C) \in \mathcal{V}_B.$$

According to the choice of  $\mathcal{V}_A$  and  $\mathcal{V}_B$  we have  $\tau(C) \in V_{\tau(A)}$  and  $\tau(\lambda_s(C)) \in V_{\tau(B)}$ , so  $\{\tau(\lambda_s(C)): s \text{ choose for } t \in G\}$  is bounded. On the other hand  $\tau(\lambda_s(C)) = s + \tau(C)$ , i.e.  $\{\tau(\lambda_s(C)): s \text{ choose for } t \in G\}$  is not bounded. This completes the proof.

\*  $V \subset G$  is bounded if there are  $p, r \in G$  such that for every  $t \in V$  we have  $p < t < r$ .

## References

- [1] N. P. Bhatia, G. P. Szegö, *Stability theory of dynamical systems*, Berlin-Heidelberg-New York, 1970.
- [2] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. 71 (1951), 152-182.
- [3] J. Kłapyta,  *$\leftarrow$ -prolongational limit sets in a generalized systems*, Zeszyty Naukowe UJ, Prace Matematyczne (w druku).
- [4] A. Pelczar, *Ogólne układy dynamiczne* (Kraków 1978).
- [5] M. Sobański, *Limit sets in generalized pseudo-dynamical systems*, Zeszyty Naukowe UJ, Prace Matematyczne t. 20, 1978.

*Received September 15, 1979*