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On the Equivalence of Whitney (a)-regularity and (a_s)-regularity

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Summary. C.T.C. Wall has conjectured in [1] that (a_s)-regularity implies (a)-regularity. This paper gives a proof of this conjecture. In [1], (a_s)-regularity is called (A'')-regularity. We will use the notations of [2]. In [2] D. J. A. Trotman gave a proof of Wall's conjecture only for semi-analytic sets. For a proof that (a)-regularity implies (a_s)-regularity see [1].

1. Let M and N be two manifolds embedded in R^n such that $N \subset \overline{M} - M$ and let $x \in N$.

Definition 1. We say that M is *(a)-regular over N at x ((M, N) satisfies the condition (a) at x)* if the following holds:
For every sequence $\{x_m\}$ of points of M tending to x such that $T_{x_m}M$ tends to τ in the Grassmannian manifold of k-dimensional subspaces of R^n ($k = \dim M$) we have $T_x N \subset \tau$.

Definition 2. We say that *M is (a_s)-regular over N at x ((M, N) satisfies the condition (a_s) at x)* if for any local C^1-retraction at x, $\Pi \subset R^n \times N$, x has a neighbourhood U such that $\Pi|_{M \cap U}$ is a submersion.

Let $G_k(n)$ denote the Grassmannian manifold of k-dimensional subspaces of R^n; $G_k(n)$ admits a structure of an analytic manifold introduced by the following atlas of inverse charts:

$$\varphi_{EF}\colon L(E, F) \ni f \to \hat{f} = \{u + f(u)\colon u \in E\} \in G_k(n),$$

where E, F are linear subspaces of R^n such that $E \oplus F = R^n$;

$$\varphi_{EF}(L(E, F)) = G'(F)$$

where $G'(F)$ denotes all algebraic supplements of F. Notice that for a base of E $\{e_i\}$, $i = 1, \dots, k$ and for $f \in L(E, F)$ the family

$$p_i(f) = e_i + f(e_i), \quad i = 1, \dots, k$$

is a base of $\hat{f}$.

2. ASSERTION 1. *Let $h \subset R^n \times R^n$ be a diffeomorphism such that* $\operatorname{dom} h$ *is open and* $x \in \operatorname{dom} h$. *Suppose that a sequence* $\{x_m\} \subset \operatorname{dom} h$ *tends to* x, $T_{x_m}M$ *tends to* τ *and* $\tau \supset T_x N$. *Then* $h(x_m)$ *tends to* $h(x)$ *and* $T_{h(x_m)}h(M)$ *tends to*:

$$d_x h(\tau) \supset T_{h(x)} h(N).$$

Proof. Since h is a diffeomorphism, $h(x_m)$ tends to $h(x)$. Choose an inverse chart φ_{EF} such that $\tau \in G'(F)$. Let $f_m \in L(E, F)$ be a sequence of linear functions such that:

$$T_{x_m}M = \hat{f}_m, \quad \tau = \hat{f}_\tau.$$

Hence the family

$$p_i(f_m), \quad i = 1, \ldots, k$$

is a base of $T_{x_m}M$ for each m. Then $p_i(f_m)$ tends to $p_i(f_\tau)$ $(i = 1, \ldots, k)$ which is a base of τ and $d_{x_m}h(p_i(f_m))$ $(i = 1, \ldots, k)$ is a base of $T_{h(x_m)}h(M)$ and tends to $d_x h(p_i(f_\tau))$ $(i = 1, \ldots, k)$ which is a base of $d_x h(\tau)$. If $\tau \supset T_x N$ then $d_x h(\tau) \supset T_{h(x)} h(N)$.

Remark 1. It is easy to see that if $\Pi \subset R^n \times N$ is a local C^1-retraction at x and x has a neighbourhood V in R^n such that $\Pi|_{M \cap V}$ is a submersion then $h(V)$ is such a neighbourhood of $h(x)$ that

$$h \circ \Pi \circ h^{-1}|_{h(M \cap V)}$$

is a submersion and

$$h \circ \Pi \circ h^{-1}|_{h(V)}$$

is a local C^1-retraction at $h(x)$.

In view of Assertion 1 if (M, N) satisfies the condition (a) (resp. (a_s)) at x, $(h(M), h(N))$ satisfies the condition (a) (resp. (a_s)) at $h(x)$. Similarly, if (M, N) does not satisfy the condition (a) (resp. (a_s)) at x, $(h(M), h(N))$ does not satisfy the condition (a) (resp. (a_s)) at $h(x)$.

3. THEOREM 1. *Whitney (a)-regularity is equivalent to (a_s)-regularity.*

Proof of the Theorem 1. It is easy to show that (a)-regularity implies (a_s)-regularity (see [1]). Now we will show that (a_s)-regularity implies (a)-regularity. Let us suppose that the condition (a) fails at x, then there exist $\{x_m\}$ a sequence of points of M tending to x such that $T_{x_m}M$ tends to τ and $\tau \not\supset T_x N$. In view of Remark 1 we can choose a diffeomorphism $h \subset R^n \times R^n$ such that $x \in \operatorname{dom} h$ and $h(N)$ is an open subset of a linear subspace of R^n. Thus we may assume that N is an open subset of a linear subspace $\hat{N} \subset R^n$ and x is the origin in $\hat{N}$. Let $e_1, \ldots, e_r, \ldots, e_k$ be a base of τ such that $e_{r+1}, \ldots, e_k$ is a base of $\tau \cap \hat{N} \subsetneq \hat{N}$ (when $\tau \cap \hat{N} \neq \{0\}$). Let us choose $e_{k+0}, \ldots, e_s$ in such a manner that $e_{r+1}, \ldots, e_s$ is a base of $\hat{N}$ (if $\tau \cap \hat{N} = \{0\}$ choose only $e_{k+1}, \ldots, e_s$ in such a manner that $e_{k+1}, \ldots, e_s$ is a base of $\hat{N}$) and let us choose $e_{s+1}, \ldots, e_n$ such that $e_1, \ldots, e_n$ is a base of R^n. We may assume that $e_1, \ldots, e_n$ is the canonical base of R^n (because we can construct a suitable isomorphism). Let $E_{e_1, \ldots, e_k}$ be span $\{e_1, \ldots, e_k\}$ and

$$L(E_{e_1, \ldots, e_k}, E_{e_{k+1}, \ldots, e_n}) \to G'(E_{e_{k+1}, \ldots, e_n})$$

the inverse chart of $G_k(n)$; $\hat{f}_m$ denotes $T_{x_m}M$. We may assume that

$$p_1(f_m)\wedge \dots \wedge p_r(f_m)\wedge e_{k+1}\wedge \dots \wedge e_n \neq 0$$

for all m (it is so for m sufficiently large). Now we will construct Π, a local C^1-retraction of R^n on a neighbourhood of the origin in $\hat{N}$, such that

$$K_m = \operatorname{span}\{p_1(f_m), \dots, p_r(f_m)\}$$

will be a subset of $\ker d_{x_m}\Pi$. In this case

$$\operatorname{im} d_{x_m}\Pi | T_{x_m}M = \operatorname{span}\{d_{x_m}\Pi(p_{r+1}(f_m)), \dots, d_{x_m}\Pi(p_k(f_m))\} \subsetneq \hat{N}$$

(submersion will fail). It is easily seen that it suffices to construct Π as a local retraction to $E_{e_{r+1},\dots,e_n} \cong R^{n-r}$ because its composition with the ortogonal projection of $E_{e_{r+1},\dots,e_n}$ on $\hat{N} = E_{e_{r+1},\dots,e_s}$ will be a local retraction of R^n to $\hat{N}$. For the required construction we will use the C^1 case of the

Whitney extension THEOREM: *Let K be a compact subset of R^n, $f_0, \dots, f_n$ a family of continuous functions to R^p. Then there exists $f \in C^1(R^n, R^p)$ such that:*

$$f|_K = f_0, \quad \left.\frac{\partial f}{\partial x_1}\right|_K = f_1, \dots, \left.\frac{\partial f}{\partial x_n}\right|_K = f_n$$

if and only if the following condition is fulfilled:

$$(*) \qquad f_0(x) = f_0(y) + f_1(y)(x_1 - y_1) + \dots + f_n(y)(x_n - y_n) + o|x-y|$$

for $x, y \in K$ and $|x-y|$ tending to the origin.

For the general proof of Whitney extension theorem see [3]. Our version is the C^1 case of the version which can be found in [4]. Let us denote

$$K = \bigcup_{m=1}^{\infty} \{x_m\} \cup X,$$

where X is a compact neighbourhood of the origin of $\{0\} \times R^{n-r} = E_{e_{r+1},\dots,e_n}$. We will construct Π such that

$$d_{x_m}\Pi = T_m,$$

where T_m is a projection of R^n on R^{n-r}, of the kernel K_m.

Let us introduce the following notations:

$$x_m = (u^m, v^m), \text{ where } u^m \in R^r, v^m \in R^{n-r}$$
$$x = (0, v) \quad \text{for } x \in X.$$

Now we construct $f_0, \dots, f_n$ as follows:

i) $f_0(o, v) = v$, $f_0(u^m, v^m) = v^m$

ii) for $i = 1, \dots, r$ $f_i(0, v) = v$, $f_i(u^m, v^m) = T_m(e_i)$.

In this case we have $T_m(p_i(f_m)) = 0$. Hence

$$T_m(f_m(e_i)) + T_m(e_i) = 0\,, \quad T_m(e_i) = -f_m(e_i) \text{ and}$$

$$\lim_{m\to\infty} T_m(e_i) = f_\tau(e_i) = 0$$

iii) for $i = r+1, \ldots, n$ $f_i(0, v) = e_{i-r} \in R^{n-r}$, $f_i(u^m, v^m) = T_m(e_i) = e_{i-r} \in R^{n-r}$.

It is easily seen that $f_0, \ldots, f_n$ are continuous on K.

In order to show that the family $f_0, \ldots, f_n$ fulfils the condition (*) of the Whitney extension theorem let us recall that for functions $f \subset R^n \times R^p$, $g \subset R^n \times R^m$ and a base of a filter $\mathscr{A}$ $f = 0_{\mathscr{A}}(g)$ if and only if for any $\varepsilon > 0$ there exists $A \in \mathscr{A}$ such that $|f(x)| \leqslant \varepsilon |g(x)|$ for all $x \in A$.

In our situation $\mathscr{A}$ is a filter base of neighbourhoods of the diagonal of K. There are four possibilities:

1° $x, y \in X$

2° $x, y \in \bigcup_{m=1}^{\infty} \{x_m\}$

3° $x \in X$, $y \in \bigcup_{m=1}^{\infty} \{x_m\}$

4° $x \in \bigcup_{m=1}^{\infty} \{x_m\}$, $y \in X$

i.e. in each case i^0 $(i = 1, 2, 3, 4)$ we will choose δ_{i^0} and then we will take

$$A = \{(x, y)\colon |x-y| < \min(\delta_{1^0}, \ldots, \delta_{4^0})\}\,.$$

Case 1°: $x = (0, v^1)$ $y = (0, v^2)$ so we have in (*)

$$v^1 = v^2 + e_1(v_1^1 - v_1^2) + \ldots + e_{n-r}(v_{n-r}^1 - v_{n-r}^2) + o|v^1 - v^2|$$

i.e.
$$0 = o|v^1 - v^2|\,.$$

Case 2°: $x = (u^p, v^p)$ $y = (u^q, v^q)$ where (u^p, v^p) and (u^q, v^q) tend to the origin, so we have in (*)

$$v^p = v^q + T_q(e_1)(u_1^p - u_1^q) + \ldots + T_q(e_r)(u_r^p - u_r^q) + e_1(v_1^p - v_1^q) + \\ + \ldots + e_{n-r}(v_{n-r}^p - v_{n-r}^q) + o|(u^p, v^p) - (u^q, v^q)|$$

i.e.
$$T_q(e_1)(u_1^p - u_1^q) + \ldots + T_q(e_r)(u_r^p - u_r^q) = o|(u^p, v^p) - (u^q, v^q)|$$

it is obvious because $T_q(e_i) \to 0$ when $q \to \infty$.

Case 3°: $x = (0, v)$ $y = (u^q, v^q)$ where (u^q, v^q) tends to the origin, so we have in (*)

$$v = v^q + T_q(e_1)(-u_1^q) + \ldots + T_q(e_r)(-u_r^q) + e_1(v_1 - v_1^q) + \ldots + \\ + e_{n-r}(v_{n-r} - v_{n-r}^q) + 0|(0, v) - (u^q, v^q)|$$

i.e.
$$T_q(e_1)(u_1^q) + \ldots + T_q(e_r)(u_r^q) = o|(0, v) - (u^q, v^q)|$$

it is obvious because $T_q(e_i) \to 0$ when $q \to \infty$.

Case 4^0: $y = (0, v)$ $x = (u^p, v^p)$ where (u^p, v^p) tends to the origin, so we have in (*)

$$v^p = v + vu_1^p + \ldots + vu_r^p + e_1(v_1^p - v_1) + \ldots + e_{n-r}(v_{n-r}^p - v_{n-r}) + \sigma |(u_p, v^p) - (0, v)|$$

i.e.

$$vu_1^p + \ldots + vu_r^p = \sigma |(u^p, v^p) - (0, v)|$$

and it is obvious as well because we may take δ_{4^0} such that if

$$|(u^p, v) - (0, v^p)| < \delta_{4^0}$$

then $|v|$ is sufficiently small.

The assumptions of the Whitney extension theorem are fulfilled, so there exists a C^1-extension of f_0, which completes the proof of Theorem 1.

References

[1] C. T. C. Wall, *Regular stratifications, Dynamical Systems Warwick 1974*, Lecture Notes in Mathematics 468, Springer, Berlin and New York, 1975, 332-344.

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The reference D.J.A. Trotman Geometric *Versions of Whitney Regularity for Smooth Stratifications*, Ann. scient. Éc. Norm. Sup., Vol. 12, 1979, pp. 461-471 came to our notice after the present paper was written. However, there are differences in adopted methods, so I hope this paper will still be of interest.

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