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The Navier-Stokes Type Equations

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Introduction. We consider in this paper some nonlinear equations in arbitrary Hilbert space. The starting point is the stationary homogeneous Navier-Stokes problem in a bounded open set Ω of $\mathbf{R}^n$, $n = 2$ or 3:

For given f, find $u = (u_1, \ldots, u_n)$ and p such that:

$$\begin{cases} \nu\Delta u - \sum\limits_{i=1}^{n} u_i D_i u = f - \operatorname{grad} p \text{ in } \Omega \\ \operatorname{div} u = 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega . \end{cases} \tag{0.1}$$

Here u and p represent the velocity and the pressure of a viscous incompressible fluid filling Ω and ν denotes the viscous coefficient.

This problem is equivalent to the functional equation (1.3) in some Hilbert space. We consider the natural generalization of (1.3). We study the existence and uniqueness problems (Sec. 2) the dependence of the solutions on F (Section 3 and 4) and on ν (Section 5). In Section 6 we give some simple examples to illustrate the theory.

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1. The formulation of the problem.

The approach presented here is based on [2]. Let Ω be a bounded domain in $\mathbf{R}^n$, $n = 2$ or 3. We denote by $C_0^\infty(\Omega)$ the space of C^∞ functions $\varphi\colon \Omega \to \mathbf{R}$ with compact support in Ω.

$$L^2(\Omega) := \{u\colon \Omega \to \mathbf{R} | u \text{ is measurable and } \int\limits_{\Omega} |u(x)|^2 dx < +\infty\}$$

$$H^1(\Omega) := \{u \in L^2(\Omega) | D_i u \in L^2(\Omega),\ i = 1, \ldots, n\}$$

where the derivatives are taken in the sense of the theory of distributions.

$H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. The scalar products in $L^2(\Omega)$, $H^1(\Omega)$, $H_0^1(\Omega)$ are denoted by

$$(u, v)_{L^2} = \int_\Omega uv, \quad (u, v)_{H^1} = (u, v)_{L^2} + \sum_{i=1}^n (D_i u, D_i v)_{L^2}$$

$$((u, v)) := \sum_{i=1}^n (D_i u, D_i v)_{L^2}$$

respectively and $||u||^2 = ((u, u))$.

If Z is a Hilbert space, Z^n denotes the space $Z \times \ldots \times Z$ (n-time) with scalar product

$$(x, y)_{Z^n} = \sum_{i=1}^n (x_i, y_i)_Z, \quad x, y \in Z^n .$$

Definition 1.1. Let $u = (u_1, \ldots, u_n)$, $u_i \in C_0^\infty(\Omega)$, $i = 1, \ldots, n$.

$$\mathscr{V} := \{u \in (C_0^\infty(\Omega))^n \mid \operatorname{div} u = 0\}$$

$$V := \text{ the closure of } \mathscr{V} \text{ in } (H_0^1(\Omega))^n .$$

We agree to use the same notation for the scalar product and the norm in V and $H_0^1(\Omega)$. We set

$$b(u, v, w) = \sum_{i,k=1}^n \int_\Omega u_k D_k v_i w_i dx \quad \text{for } u, v, w \in V.$$

It is well known that: (see [1], [2])

(1.1) b: $V \times V \times V \to \mathbf{R}$ is trilinear and continuous i.e. there exists $c>0$ such that $|b(u, v, w)| \leqslant c||u|| \cdot ||v|| \cdot ||w||$

and that $b(u, v, w) = -b(u, w, v)$ for all $u, v, w \in V$.

Definition 1.2. Let $f \in V'$ (the dual space of V). We say that u is the *weak solution* of (0.1) $u \in V$ and

$$\nu((u, v)) = b(u, v, w) + f(v) \text{ for all } v \in V. \tag{1.2}$$

For any $f \in V'$ applying the Riesz theorem we get a unique $F \in V$ satisfying

$$((F, v)) = f(v) \quad \text{for all } v \in V.$$

From (1.1) we know that $b(u, \cdot, w)$: $V \to \mathbf{R}$ is a linear continuous form on V. By Riesz theorem, there exists an element of V, (we denote it by $B(u, w)$) such that

$$b(u, v, w) = ((B(u, w), v)) \quad \text{for all } v \in V.$$

We write $Bu := B(u, u)$. The operator B: $V \to V$ is continuous (see (1.1)). One can prove (see [2]) that it is a compact operator. Now we can write the equation (1.2) in the form

$$\nu((u, v)) = ((Bu, v)) + ((F, v)) \quad \text{for all } v \in V.$$

Hence we have

$$\nu u = Bu+F . \tag{1.3}$$

The operator $N_\nu = \nu E - B$ is called the abstract operator of Navier-Stokes (see [2], [6]).

We consider the following generalization of the equation (1.3).

Definition 1.3. Let X be an arbitrary Hilbert space with the scalar product $(\cdot, \cdot)$ and let $B(\cdot, \cdot)\colon X \times X \to X$ be a bilinear operator such that

1°. $(B(u, w), v) = -(B(u, v), w)$ for all $u, v, w \in X$,

2°. $B(\cdot, \cdot)\colon X \times X \to X$ is continuous i.e. there exists $c>0$ such that $\|B(u, v)\| \leqslant c\|u\| \cdot \|v\|$ for all $u, v \in X$,

3°. The operator $B\colon u \to Bu := B(u, u)$ is compact.

By the above assumptions the equation

$$\nu u = Bu+F$$

is called the *equation of Navier-Stokes type* in the space X (shortly: of *N-S* type).

It is a natural generalization of (1.3). An analogous generalization has been considered in [11] (cf. also [8]).

We shall study this type of equations in the sequel. The following simple properties of the operator B will be often used.

$$(B(v, u), u) = 0, \text{ in particular } (Bu, u) = 0 \tag{1.4}$$

$$B(u \pm v) = Bu + Bv \pm B(u, v) \pm B(v, u) \tag{1.5}$$

$$B(tu) = t^2 Bu,\ t \in \mathbf{R} \tag{1.6}$$

$$\|Bu\| \leqslant c\|u\|^2 . \tag{1.7}$$

2. The existence and uniqueness problems.

THEOREM 2.1. a) *For every* $\nu>0$ *and* $F \in X$ *there exists a solution of the equation*

$$\nu u = Bu+F \tag{2.1}$$

b) *If* u *is a solution of* (2.1) *then*

$$\|u\| \leqslant \frac{\|F\|}{\nu} \quad \text{and} \tag{2.2}$$

$$\|u\| > \frac{F}{\sqrt{\nu^2+\delta^2}} \quad \text{for } F \neq 0 \text{ where } \delta := \sqrt{c\|F\|} . \tag{2.3.}$$

Proof: a) The existence of solution of (2.1) follows directly from the Leray-Schauder-Schaefer theorem (see [5], [3], cf. also [2]). An essential step in the proof is to obtain the a priori inequality (2.2).

b). We take the scalar product of (2.1) with u. Using (1.4) we get

$$\nu\|u\|^2 = (F, u) \leqslant \|F\|\cdot\|u\| \text{ and hence (2.2)}.$$

Now we are going to prove (2.3). We take the scalar product of (2.1) with Bu. Using (1.4) and (1.7) we get

$$\|Bu\|^2 = -(F, Bu) \leqslant \|F\|\cdot\|Bu\| \leqslant \|F\|\cdot c\cdot\|u\|^2 \quad \text{i.e.}$$

$$\|Bu\| \leqslant \delta\|u\| \quad \text{where } \delta = \sqrt{c\|F\|}\,. \tag{2.4}$$

From (2.1) we have

$$\|F\|^2 = \|\nu u - Bu\|^2 = \nu^2\|u\|^2 + \|Bu\|^2\,.$$

Hence and from (2.4) we obtain

$$\|F\|^2 \leqslant \nu^2\|u\|^2 + \delta^2\|u\|^2 \quad \text{thus } \|u\| \geqslant \frac{\|F\|}{\sqrt{\nu^2+\delta^2}}\,.$$

It is easy to see that the equality in the above inequality can take place only if $F = 0$

THEOREM 2.2. *If $\nu > \delta$ then the equation* (2.1) *has exactly one solution.*

Proof. The proof is standard (cf. [2], [1]).

Let B'_u denote the Fréchet derivative of the operator B in the point u. It is easy to see that B'_u exists for each u and

$$B'_u(\cdot) = B(u, \cdot) + B(\cdot, u)\,. \tag{2.5}$$

We have the following

LEMMA 2.1. *Either*

(A) *for all $\nu > 0$ and for all $u \in X$ the operator $\nu E - B'_u$ is invertable or*

(B) *for every $\nu > 0$ there exists $F \in X$ such that the equation $\nu x - Bx = F$ has at least two solutions.*

Proof: 1° ($\sim$(A) $\Rightarrow$ (B)). Suppose that (B) is not true i.e. there exist $\nu_0 > 0$ and $u_0 \in X$ such that the operator $\nu_0 E - B_{u_0}$ is not invertable. Thus ν_0 is an eigenvalue of B'_{u_0}, i.e. there exists $\varphi \in X$, $\|\varphi\| = 1$ such that

$$\nu_0\varphi = B'_{u_0}(\varphi)\,. \tag{2.6}$$

From (1.5), (2.5) and (2.6) we get

$$\begin{aligned} \nu_0(u_0+\varphi) - B(u_0+\varphi) &= \nu_0 u_0 - Bu_0 - B\varphi \\ \nu_0(u_0-\varphi) - B(u_0-\varphi) &= \nu_0 u_0 - Bu_0 - B\varphi\,. \end{aligned} \tag{2.7}$$

Let $v_1 = u_0 + \varphi$

$v_2 := u_0 - \varphi$

$F_1 := \nu_0 u_0 - Bu_0 - B\varphi$.

Using (2.7) we obtain

$$\nu_0 v_i - Bv_i = F_0,\ i = 1,2 \text{ and } v_1 \neq v_2 .$$

Substituting $v_i = \frac{\nu_0}{\nu} x_i$, $i = 1,2$, and using (1.6) we get

$$\nu x_i - Bx_i = F \quad \text{where } F := \frac{\nu^2}{\nu_0^2} F_1$$

and 1° is proved.

2°. ((B) $\Rightarrow$ $\sim$(A)). There exist $\nu_0 > 0$ and $F_0 \in X$ such that the equation $\nu_0 u - Bu = F_0$ has two different solutions u_1 and u_2.

We put $x = \frac{u_1 + u_2}{2}$, $y = \frac{u_1 - u_2}{2}$. Then we have

$$\nu_0(x+y) = B(x+y) + F_0$$
$$\nu_0(x-y) = B(x-y) + F_0 .$$

Subtracting the above equalities and using (1.5) and (2.5) we get

$$\nu_0 y = B'_x(y) .$$

Since $y \neq 0$, the proof is complete.

THEOREM 2.3. (A) $\Leftrightarrow$ $\forall \nu > 0$, $\forall F \in X$ *the equation* $\nu u - Bu = F$ *has exactly one solution.*

Proof: ($\Leftarrow$). Suppose that (*A*) does not hold. Hence, by Lemma 2.1, we obtain the contradiction with the assumption.

($\Rightarrow$). It is, in essence, a particular case of step 2° of the proof of Lemma 2.1.

Remark. The theorem analogous to Th. 2.3 is proved in [11]. The proof presented here is more elementary.

3. The dependence of the solutions on F.

THEOREM 3.1. *Let* u_n, u *be the solutions of the equation* $\nu x - Bx = g$ *with the right-hand sides* $g = F_n$, $g = F$ *respectively. Assume that*

$$c\|F\| < \nu^2. \tag{3.1}$$

If $F_n \to F$ *then* $u_n \to u$ *as* $n \to \infty$.

Remark. It follows from Theorem 2.2 that for n sufficiently large u_n, u are the unique solutions.

Proof: We have

$$\nu u_n = Bu_n + F_n$$
$$\nu u = Bu + F .$$

We take the scalar product of the first equality with u and of the second equality with u_n. Using (1.4) we obtain

$$\nu(u_n, u) = (Bu_n, u) + (F_n - F, u) + \nu\|u\|^2$$

$$\nu(u, u_n) = (Bu, u_n) + (F - F_n, u_n) + \nu\|u_n\|^2 .$$

Let $w := u - u_n$, $W := F - F_n$. Adding the above equalities and using the fact that

$$(B(u-u'), u) = (Bu', u) + (Bu, u') \quad \text{for all } u, u' \in X,$$

we obtain

$$\nu\|w\|^2 = -(Bw, u) + (W, w).$$

From (2.2) we get

$$\nu\|w\|^2 \leqslant c\|w\|^2\|u\| + \|W\|\,\|w\| \leqslant c\|w\|^2 \frac{\|F\|}{\nu} + \|W\|\,\|w\| .$$

Hence

$$\|w\|^2(\nu^2 - c\|F\|) \leqslant \nu\|w\|\,\|W\| .$$

The assumption 3.1 allows us to write

$$\|w\| \leqslant \frac{\nu\|W\|}{\nu^2 - c\|F\|} .$$

Finally

$$\|u_n - u\| \leqslant K_1\|F_n - F\| \quad \text{where } K_1 > 0 .$$

The proof is completed.

We need a few definitions to formulate the next theorem.

Definition 3.1. (cf. [6]) Let X and Y be two Banach spaces. An operator $L \in \mathscr{L}(X, Y)$ is called a *Fredholm operator* if

b) $L(X)$ is closed

b) The spaces $\operatorname{Ker} L$ and $\operatorname{Coker} L = Y/L(X)$ have finite dimensions.

If L is Fredholm, then its index is the integer equal to

$$\dim \operatorname{Ker} L - \dim \operatorname{Coker} L .$$

LEMMA 3.1. (cf. [6]) *The operator of the form $E - K$, where $K \in \mathscr{L}(X, X)$ is compact, is a Fredholm operator and*

$$\operatorname{ind}(E - K) = 0 .$$

Definition 3.1. (cf. [6]) A nonlinear operator $N \in C^1(X, Y)$ is called a *Fredholm operator* if N'_x, its Fréchet differential in the point x, is a Fredholm operator. We put $\operatorname{ind} N := \operatorname{ind} N_x$. One can prove that $\operatorname{ind} N_x$ is independent of x (cf. [6]).

Definition 3.2. Let $N \in C^1(X, Y)$. A point $u \in X$ is called a *regular point* of N if N'_u is surjective and a singular point if it is not regular. The images of the singular points under N are called the critical values.

We have the following

THEOREM 3.2. (Smale, cf. [6]) *Let $N\colon X \to Y$ be a Fredholm operator of class C^q. If $q > \max(\operatorname{ind} N, 0)$ then the set of all critical values of N is nowhere dense in Y.*

Let $S(v, F)$ denote the set of all solutions of the equation

$$N_v u = F \tag{3.2}$$

where $N_v = vE - B$.

The following theorem gives a generic property of $S(v, F)$ (cf. [8]).

THEOREM 3.3. *For every fixed $v > 0$*

(a) *there exists a dense open set $O(v) \subset X$ such that, for every $F \in O(v)$, the set $S(v, F)$ is finite.*

(b) *For each connected component $O_s(v)$ of $O(v)$ the number of points in $S(v, F)$ for $F \in O_s(v)$ is constant.*

(c) *Every solution is a C^∞ function of F for $F \in O_s(v)$.*

Proof: It is easy to see that the operator N_v is of class C^∞ (see (2.5)). We have

$$N'_v(x) = vE - B'_x .$$

Since the operator B is compact, the operator B'_x is also compact (see [10], p. 55). From Lemma 3.1 we obtain that N_v is the Fredholm operator of class C^∞ and $\operatorname{ind} N_v = 0$ (cf. Definitions 3.1 and (3.1)).

Let $K(v)$ denote the set of all critical values of N_v. We put

$$O(v) := X \setminus K(v) . \tag{3.3}$$

It follows from the Smale theorem that $O(v)$ is dense in X. The other properties of $O(v)$ can be proved in the same way as in [8].

COROLLARY 3.1. *Let $K(0, r) := \{F \in X \mid \|F\| < r\}$ and let $r < \frac{v^2}{c}$. The equation $vx - Bx = F$ has the unique solution u for $F \in K(0, r)$* (by Theorem 2.2) *and the function $u = u(F)$ s of class C^∞.*

Proof: We shall prove that if $F \in K(0, r)$, then F is a regular value of the operator $N_v = vE - B$. Indeed, suppose F is a critical value of N_v. Then the solution u is a singular point of N_v and it is easy to see that v is the eigenvalue of the operator B'_u. So there exists $\varphi \in X$, $\|\varphi\| \neq 0$, such that

$$v\varphi = B'_u(\varphi) = B(u, \varphi) + B(\varphi, u) .$$