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Remark on the Chaplygin Method for a System of First Order Partial Differential-functional Equations

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Abstract. In this paper the Cauchy problem for the following system of first order partial differential equations is considered

$$\frac{\partial z^r(x,y)}{\partial x}+\sum_{j=1}^{n} a_j^r(x,y)\frac{\partial z^r(x,y)}{\partial y_j}=f^r(x,y,z^r(x,y),z),\ r\in I \tag{0.1}$$

in the zone

$$W_\alpha=\{(x,y)\colon\ x\in \mathbf{R},\ 0\leqslant x<\alpha,\ y\in \mathbf{R}^n\}$$

where the set of indexes I is arbitrary, that means it need not be finite or countable. For solution of this system the Chaplygin method is applied. Theorem 2.1 gives a construction of a Chaplygin sequence and Theorem 2.2 establishes some properties of this sequence, like its monotony and convergence to the regular solution of system (0.1). The results obtained are a generalization of those given in the paper [1].

1. Notations, definitions and assumptions

We shall use the following notations: if $y=(y_1,\ldots,y_n)\in\mathbf{R}^n$ then

$$|y|=\max\{|y_i|\colon\ i=1,\ldots,n\}.$$

Similarly: if $Q=\{q_{ij}\}_{i,j=1,\ldots,n}$ is $n\times n$ matrix, then

$$|Q|=\max\{|q_{ij}|\colon\ i,j=1,\ldots,n\}.$$

Definition 1.1. Let I be an arbitrary, not empty subset of $\mathbf{R}$. By $\mathscr{F}(I)$ we denote the Banach space of real functions defined and bounded in I with the norm

$$\|w\|_{\mathscr{F}}=\sup\{|w(r)|\colon\ r\in I\}.$$

The value of the function $w\in\mathscr{F}(I)$ in a point $r\in I$ will be denoted by the symbol w^r.

For a fixed $\alpha>0$ let

$$W_\alpha=\{(x,y)\in\mathbf{R}^{n+1}\colon\ 0\leqslant x<\alpha,\ y\in\mathbf{R}^n\}.$$

Definition 1.2. Let $\mathscr{X}_\alpha$ denote the Banach space of mappings

$$z\colon\ W_\alpha \in (x, y) \to z(x, y) \in \mathscr{F}(I)$$

where for every fixed $r \in I$ the function

$$z^r\colon\ W_\alpha \ni (x, y) \to z^r(x, y) \in \mathbf{R}$$

is bounded and continuous in W_α, with the norm

$$\|z\| = \sup\{\|z(x, y)\|_{\mathscr{F}}\colon\ (x, y) \in W_\alpha\}\,.$$

If the set I is finite then z^r is the r-th coordinate of the function z.

In the space $\mathscr{X}_\alpha$ we also consider the following norm: if $\tilde{x}$ is a fixed number such that $0 \leqslant \tilde{x} < \alpha$, then for any fixed $z \in \mathscr{X}_\alpha$ we set

$$\|z\|_{\tilde{x}} = \sup\{\|z(x, y)\|_{\mathscr{F}}\colon\ 0 \leqslant x \leqslant \tilde{x},\ (x, y) \in W_\alpha\}\,.$$

For a fixed $r \in I$ the symbol $\mathscr{D}_r$ will denote the differential operator of the form

$$\mathscr{D}_r = \frac{\partial}{\partial x} + \sum_{j=1}^{n} a_j^r(x, y)\frac{\partial}{\partial y_j}\,.$$

We shall consider the following system of semi-linear partial differential-functional equations

$$\mathscr{D}_r z^r(x, y) = f^r(x, y, z^r(x, y), z),\ \ r \in I \tag{1.1}$$

where for every fixed $r \in I$ the real function f^r is defined in the set $W_\alpha \times \mathbf{R} \times \mathscr{X}_\alpha$.

Definition 1.3. A mapping $z \in \mathscr{X}_\alpha$ is said to be a regular solution of system (1.1) in the zone W_α if for every fixed $r \in I$ the function $z^r\colon W_\alpha \ni (x, y) \to z^r(x, y) \in \mathbf{R}$ is of class C^1 in W_α and the equations of system (1.1) are satisfied in W_α.

As in the paper [1], in order to simplify the formulation of theorems we introduce also the following definition

Definition 1.4. A real function v defined in W_α is called the function of class $S(A; K, C, L_1, W_\alpha)$ if it is of class $C^1(W_\alpha)$ and the following inequalities hold true

$$|v(x, y)| \leqslant D$$

$$\left|\frac{\partial v(x, y)}{\partial y}\right| \leqslant D_1$$

$$\left|\frac{\partial v(x, y)}{\partial x}\right| \leqslant D_2$$

$$\left|\frac{\partial v(x, y)}{\partial y} - \frac{\partial v(\bar{x}, \bar{y})}{\partial y}\right| \leqslant L^{**}|x - \bar{x}| + L^{*}|y - \bar{y}|$$

for $(x, y), (\bar{x}, \bar{y}) \in W_\alpha$, where

$$D = e(K+e-1) \tag{1.2}$$

$$D_1 = eK+nCe^{2n+1}+3e^2 \tag{1.3}$$

$$D_2 = Ae(K+n^2e^{2n}C+2e-1) \tag{1.4}$$

$$L^* = e[2K+2ne^{2n}(1+n^4e^{4n})C+n^2e^{4n}L_1+11e-2] \tag{1.5}$$

$$L^{**} = Ae[(1+A)(K+4e)+ne^{2n}(A+4n^4e^{4n})C+n^2e^{4n}L_1] \tag{1.6}$$

and $\dfrac{\partial v}{\partial y} = \left(\dfrac{\partial v}{\partial y_1}, \ldots, \dfrac{\partial v}{\partial y_n}\right)$.

Now, we shall formulate the essential assumptions admitted in this paper.

ASSUMPTIONS H_1. *For every fixed* $r \in I$

1° a) *function* $a^r = (a^r_1, \ldots, a^r_n)$ *is defined and continuous in the zone* W_α,

b) *derivatives* $\dfrac{\partial a^r}{\partial y} = \left\{\dfrac{\partial a^r_j}{\partial y_k}\right\}_{j,k=1,\ldots,n}$ *are continuous in* W_α,

c) *derivatives* $\dfrac{\partial a^r}{\partial y}$ *are L-lipschitzian in* y.

2° *real function* $f^r(x, y, s, w)$ *is defined and continuous for* $(x, y) \in W_\alpha$, $s \in \mathbf{R}$, $w \in \mathscr{X}_\alpha$.

3° *partial derivatives* $f^r_{y_k}, f^r_{sy_k}$ $(k = 1, \ldots, n)$, f^r_{ss} *are continuous with regard to all their variables.*

4° *functions* a^r, f^r *and their derivatives* $\dfrac{\partial a^r}{\partial y}, f^r_{y_k}, f^r_s, f^r_{sy_k}, f^r_{ss}$ *are bounded by a constant B* $(k = 1, \ldots, n)$.

5° *derivatives* $f^r_{y_k}, f^r_{sy_k}, f^r_s, f^r_{ss}$ *are L-lipschitzian in y and s* $(k = 1, \ldots, n)$

6° *a function* $p\colon \mathbf{R}^n \ni y \to p(y) \in \mathscr{F}(I)$ *is given such that for every fixed* $r \in I$ *the function* $p^r\colon \mathbf{R}^n \ni y \to p^r(y) \in \mathbf{R}$ *is of class* C^1 *in* $\mathbf{R}^n$ *and for* $\bar{y}, y \in \mathbf{R}^n$

$$|p^r(y)| \leqslant K$$

$$\left|\frac{\partial p^r(y)}{\partial y}\right| \leqslant C$$

$$\left|\frac{\partial p^r(y)}{\partial y} - \frac{\partial p^r(\bar{y})}{\partial y}\right| \leqslant L_1|y-\bar{y}|$$

where K, C, L *are constant,*

7° *under the following notations*

$$A = B[1+D(1+D_1)]$$

$$\beta = \min\left\{\alpha, \frac{1}{A}, [L(1+nD_1)^3+B(DL^*+nD_1^2)]^{-1}\right\}$$

$$W_\beta = \{(x, y) \in W_\alpha\colon 0 \leqslant x < \beta\}$$

where D, D_1, L^* are defined by (1.2), (1.3) *and* (1.5) *respectively, a mapping $u \in \mathscr{X}_\beta$ is given, defined in the zone W_β, such that for every fixed $r \in I$ the function $u^r\colon W_\beta \ni (x, y) \to u^r(x, y) \in \mathbf{R}$ is of class $S(A; K, C, L_1, W_\beta)$ and*

$$u(0, y) = p(y) \quad \text{for } y \in \mathbf{R}^n.$$

ASSUMPTIONS H_2. *For every fixed $r \in I$*

1° a) *functions a^r, f^r and derivative f^r_s are L-lipschitzian in x*

b) *function f^r satisfies the following Lipschitz condition in w*

$$|f^r(x, y, s, w) - f^r(x, y, s, \bar{w})| \leqslant L\|w - \bar{w}\|_x$$

2° a) *function f^r is increasing in w*

b) *function f_r^s is increasing in s*

3° *the mapping $u \in \mathscr{X}_\beta$, defined in the point 7° of Assumption* H_1, *satisfies in W_β the following system of the differential-functional inequalities*

$$\mathscr{D}_r u^r(x, y) \leqslant f^r(x, y, u^r(x, y), u), \quad r \in I.$$

(All constants used in Assumptions H_1 and H_2 are independent of $r \in I$).

Remark 1. Let us notice that from the point 1° b) of Assumptions H_2 it follows that f^r is a functional of Volterra type in w.

Remark 2. If, for a fixed r, the right-hand side of the system (1.1) depends on the value of the function $z^r(x, y)$ for some other $r \in I$, then this dependence can be included to the functional dependence.

2. The Chaplygin method

Theorem 2.1. *Under Assumptions* H_1 *there exists a sequence of mappings $z_m \in \mathscr{X}_\beta$ ($m = 0, 1, \ldots$) such that for any fixed $m \in N$ and $r \in I$ function $z^r_m\colon W_\beta \ni (x, y) \to z^r_m(x, y) \in \mathbf{R}$ is a solution of class $S(A; K, C, L_1, W_\beta)$ of the linear partial differential equation*

$$\mathscr{D}_r z^r_m(x, y) = f^r(x, y, z^r_{m-1}(x, y), z_{m-1}) + f^r_s(x, y, z^r_{m-1}(x, y), z_{m-1})\big(z^r_m(x, y) - z^r_{m-1}(x, y)\big)$$

satisfying the initial condition

$$z^r_m(0, y) = p^r(y) \quad \text{for } y \in \mathbf{R}^n$$

where

$$z_0(x, y) = u(x, y).$$

Proof. Let us fix an $r \in I$ and consider the equation

$$\mathscr{D}_r z^r(x, y) = f^r(x, y, u^r(x, y), u) + f^r_s(x, y, u^r(x, y), u)\big(z^r(x, y) - u^r(x, y)\big) \tag{2.1}$$

with the initial condition

$$z^r(0, y) = p^r(y) \quad \text{for } y \in \mathbf{R}^n. \tag{2.2}$$

As in the proof of Theorem 1 in the paper [1] we can show that the coefficients of the equations (2.1) satisfy the assumptions of Lemma 2 of that paper. Therefore the problem (2.1), (2.2) has exactly one solution z^r of class $S(A;\,K,C,L_1,W_\beta)$ in W_β. Thus we obtain the mapping $z\in\mathscr{X}_\beta$, where for every fixed $r\in I$ the function $z^r\colon W_\beta\ni(x,y)\to\to z^r(x,y)\in\mathbf{R}$ is the solution of class $S(A;\,K,C,L_1,W_\beta)$ of the problem (2.1), (2.2). Moreover the mapping z has the same properties as the mapping u in the point 7° of Assumptions H_1. The above reasoning defines the transformation T such that

$$T(u)=z$$

and it proves that if we set

$$z_m=T^m(u)$$

then the sequence $\{z_m\}$ is well defined.

The sequence $\{z_m\}$ will be called the Chaplygin sequence. The next theorem deals with some properties of this sequence and shows its convergence to a solution of the system (1.1).

THEOREM 2.2. *Under the Assumptions* H_1 *and* H_2 *the Chaplygin sequence* $\{z_m\}$ *has the following properties*:

1° *for every fixed* $m\in N$ z_m *satisfies the following system of the differential-functional inequalities*

$$\mathscr{D}_r z_m^r(x,y)\leqslant f^r\big(x,y,z_m^r(x,y),z_m\big),\quad r\in I$$

in the zone W_β

2° *the sequence* $\{z_m\}$ *is increasing*

$$u(x,y)\leqslant z_m(x,y)\leqslant z_{m+1}(x,y),\quad (x,y)\in W_\beta$$

3° *there exists a mapping* $z\in\mathscr{X}_\beta$ *such that if* $m\to+\infty$ *the sequence* $\{z_m\}$ *converges to* z *in the norm of the space* $\mathscr{X}_\beta$ *and* z *is the regular solution of the system* (1.1) *satisfying the initial condition*

$$z(0,y)=p(y)\quad \textit{for}\ y\in\mathbf{R}^n. \tag{2.3}$$

Moreover, for every fixed $r\in I$ *the function* $z^r\colon W_\beta\ni(x,y)\to z^r(x,y)\in\mathbf{R}$ *is of class* $S(A;\,K,C,L_1,W_\beta)$ *and the following inequality is fulfilled*

$$\|z(x,y)-z_m(x,y)\|_{\mathscr{X}}\leqslant 2D\frac{[(2B+L)e^2]^{m-1}x^{m-1}}{(m-1)!}\quad \textit{for}\ 0\leqslant x<\beta. \tag{2.4}$$

Proof. As in the proof of Theorem 2 in [1] we can show (by induction) the points 1° and 2° using first order partial differential weak inequalities theorem (Theorem 59.1 in [2]).

Since the functions $z_m^r(m\in N)$ are of class $S(A;\,K,C,L_1,W_\beta)$, they are equibounded in W_β. It follows from the monotony of the sequence $\{z_m^r(x,y)\}$ in W_β that for every fixed $r\in I$ the sequence $\{z_m^r(x,y)\}$ is convergent for every $(x,y)\in W_\beta$. So there exist functions z^r defined in W_β such that

$$\lim_{m\to\infty} z_m^r(x,y)=z^r(x,y),\quad r\in I$$

Next, by the same argument as was used in the proof of Theorem 2 in [1], we can show the following inequality

$$\|z_m(x,y)-z_{m-1}(x,y)\|_{\mathcal{F}} \leqslant 2D\,\frac{[(2B+L)e^2]^{m-1}x^{m-1}}{(m-1)!} \tag{2.5}$$

for every $(x, y) \in W_\beta$. This implies that the Chaplygin sequence satisfies the Cauchy condition in W_β, so it converges to z in the norm of the space $\mathcal{X}_\beta$.

On the other hand, using Arzela's theorem we can show that for every fixed $r \in I$ there exist a subsequence $\{z^r_{m_\nu}\}$ of the sequence $\{z^r_m\}$ such that

$$\lim_{\partial\to\infty} \frac{\partial z^r_{m_\nu}(x,y)}{\partial y_k} = \frac{\partial z^r(x,y)}{\partial y_k}, \qquad k = 1,\dots,n$$

$$\lim_{\nu\to\infty} \frac{\partial z^r_{m_\nu}(x,y)}{\partial x} = \frac{\partial z^r(x,y)}{\partial x}$$

and these convergences are uniform in W_β. Hence, among other, it follows that the function z^r is of class $S(A;\, K, C, L_1, W_\beta)$. By passing to the limit in equations which are satisfied by the functions z^r_m we conclude that the mapping z is a regular solution of the system (1.1) satisfying the initial condition (2.3).

The proof of the inequality (2.4) is similar to the proof of the inequality (2.5).

References

[1] M. Nowotarska, *Chaplygin method for an infinite system of first order partial differential-functional equations*, Zeszyty Naukowe UJ, Prace Matematyczne 22 (1981), 125–142.

[2] J. Szarski, *Differential Inequalities*, Warszawa 1965.

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