UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXV

1985

Highly Noncontinuable Functions on Polynomially Convex Sets

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Abstract. By the method of the extremal function Φ_K (for the definition and properties of Φ_K see [12], [15], [16]) for each polynomially convex compact subset K of C^N we construct a continuous quasianalytic function φ: $K \to C$ that is holomorphic in the interior of K and cannot be continued analytically from K in a strong sense. Moreover, we show that if D is a bounded convex domain in C^N and $K = \bar{D}$ is the topological closure of D, then the function φ is $\mathscr{C}^\infty$ on K and for every complex line L the plane set $L \cap D$ is the maximal domain of existence of the holomorphic function $\varphi|L \cap D$. The same is true if D is a bounded domain with Lipschitz boundary and $K = \bar{D}$ is polynomially convex. Thus we get the affirmative answer to questions asked by N. Sibony, Globevnik-Stout [3] and W. Rudin ([11], p. 415).

1. Introduction.

Globevnik and Stout [3] proved the following theorem:

If D is a bounded strictly convex domain in C^N with a $\mathscr{C}^2$ boundary, then there exists a holomorphic function φ in the Nevanlinna class on D with the following "high noncontinuation property":

For every complex line L the plane domain $L \cap D$ is the natural domain of existence of $\varphi|L \cap D$.

This theorem gives an affirmative answer to a question asked by N. Sibony. Globevnik and Stout [3] and W. Rudin ([11], p. 415) asked if it is possible to construct such a noncontinuable function with continuous or smooth boundary values. Their technique could not yield such functions. In this paper we answer their question in the affirmative by the method of the extremal function Φ_K ([12], [15], [16]). Let us recall that for every compact subset K of C^N the extremal function Φ_K is defined by the formula

$$\Phi_K(x) := \sup_{n \geq 1} (\sup\{|P(x)|, P \in \mathscr{P}_n, ||P||_K \leqslant 1\})^{1/n}, \quad x \in C^N, \tag{1}$$

where $\mathscr{P}_n = \mathscr{P}_n(C^N, C)$ is the set of all complex-valued polynomials of N complex variables of degree at most n, and

$$||P||_K := \sup\{|P(x)|, x \in K\}$$

is the supremum norm of P on K.

We shall need the following properties of Φ_K (see [12], [15]).

1.1. **(Bernstein-Walsh inequality).**

$$|P(x)| \leqslant ||P||_K \Phi_K^n(x), \ x \in C^N, \ P \in \mathscr{P}_n(C^N, C).$$

1.2. $\Phi_K = \Phi_{\hat{K}}$, where $\hat{K}$ is the polynomially convex envelope of K.

1.3. $\Phi_K(x) = 1$ on $\hat{K}$, $\Phi_K(x) > 1$ on $C^N - \hat{K}$.

1.4. $\Phi_{K_1} \leqslant \Phi_{K_2}$ in C^N, if $K_2 \subset K_1$.

1.5. Φ_K is locally bounded in C^N if and only if K is not pluripolar.

Let us recall that a subset E of C^N is called *pluripolar* if there exists a plurisubharmonic function W in C^N such that $W = -\infty$ on E.

1.6. If K_j is a compact subset of C^{N_j} $(j = 1, 2)$ then

$$\Phi_{K_1 \times K_2}(x, y) = \max\{\Phi_{K_1}(x), \Phi_{K_2}(y)\}, \ (x, y) \in C^{N_1} \times C^{N_2}.$$

1.7. If $N = 1$ and $K \subset C^1 = C$ is a compact set with positive logarithmic capacity then $\log \Phi_K$ is the Green function of $C - \hat{K}$ with pole at ∞.

1.8. If $||\cdot||$ is any norm in C^N and $B(a, r) = \{x \in C^N, \ ||x-a|| \leqslant r\}$ is a closed ball, then

$$\Phi_{B(a,r)}(x) = \max\{1, ||x-a||/r\}, \ x \in C^N.$$

The crucial role in our considerations is played by the following two Lemmas.

LEMMA 1. *If K is a compact subset of C^N then there exist an increasing sequence of positive integers $(n_j)_{j \geqslant 1}$ and a sequence of polynomials $(P_j)_{j \geqslant 1}$ of N complex variables such that*

(i) $\lim\limits_{j \to \infty} (\sqrt{n_{j+1}}/n_j) = +\infty$,

(ii) $\deg P_j \leqslant n_j$,

(iii) $\Phi_K(x) = \sup\limits_j |P_j(x)|^{1/n_j} = \limsup\limits_{j \to \infty} |P_j(x)|^{1/n_j}, \ x \in C^N$.

LEMMA 2. (An extension of the Ostrowski theorem on lacunary power series to the case of series of polynomials of N complex variables with lacunary sequence of degrees, see [2], [5], [6], [15]).

Let f be a holomorphic function in a domain $D \subset C^N$ and let E be a nonpluripolar compact subset of D. Assume that $(n_j)_{j \geqslant 1}$ is an increasing sequence of positive integers and let $(P_j)_{j \geqslant 1}$ be a sequence of polynomials of N complex variables such that

(a) $\deg P_j \leqslant n_j$,

(b) $\lim\limits_{s \to \infty} ||f - \sum\limits_{j=1}^{s} P_j||_E^{1/n_s} = 0$.

Then for every compact subset F of D

$$\lim ||f - \sum_{j=1}^{s} P_j||_F^{1/n_s} = 0;$$

in particular

$$f(x) = \sum_{j=1}^{\infty} P_j(x) \quad \textit{for all } x \textit{ in } D.$$

Moreover, if Ω denotes the maximal open subset of C^N in which the series $\sum P_j$ is locally uniformly convergent, then the maximal domain of existence S of f is identical with the connected component of Ω containing D. In particular S is univalent.

Now, if K is a polynomially convex compact subset of C^N, the required function $\varphi: K \to C$ with a strong non-continuation property may be defined by the formula

$$(\S) \qquad \varphi(x) = \sum_{j=1}^{\infty} \Theta^{\sqrt{n_j}} P_j(x), \quad x \in K,$$

where Θ is a fixed real number with $0<\Theta<1$, and $(n_j)_{j\geq 1}$ and $(P_j)_{j\geq 1}$ are sequences satisfying the statements (i), (ii) and (iii) of Lemma 1.

It is clear that the series (§) is uniformly convergent on K, because $\|P_j\|_K \leq 1$ and the number series $\sum_{j=1}^{\infty} \Theta^{\sqrt{n_j}}$ is convergent; the function φ is continuous on K and holomorphic in the interior $\mathring{K}$ of K. The series (§) is divergent at each point x of $C^N - K$, because

$$\limsup_{j\to\infty} (\Theta^{\sqrt{n_j}} |P_j(x)|)^{1/n_j} = \Phi_K(x) > 1 .$$

Observe that

$$\|\varphi - \sum_{j=1}^{s} \Theta^{\sqrt{n_j}} P_j\|_K \leq \sum_{j=s+1}^{\infty} \Theta^{\sqrt{n_j}} \leq M \Theta^{\sqrt{n_{s+1}}}, \; M = \text{const}.$$

Therefore

$$(+) \qquad \lim_{s\to\infty} \|\varphi - \sum_{j=1}^{s} \Theta^{\sqrt{n_j}} P_j\|_K^{1/n_s} = 0 .$$

Hence as a direct consequence of Lemmas 1 and 2 we get the following main result of this paper

THEOREM 1. *If K is a polynomially convex compact subset of C^N then the function φ defined by* (§) *has the following strong noncontinuation property*:

(SNCP) *If $\gamma: C^M \to C^N$ is any polynomial mapping of C^M into C^N with* $\deg\gamma \geq 1$, $M \geq 1$, *and if h is a holomorphic function on a ball $B(a, r) \subset C^M$ with center a and radius r such that $h = \varphi \circ \gamma$ on a nonpluripolar compact subset E of the set $F := \gamma^{-1}(K)$, then $B(a, r) \subset F$. In particular $\varphi \circ \gamma$ has no analytic continuation from the interior $\mathring{F}$ of F.*

Moreover $\varphi \circ \gamma$ is quasientire according to the terminology of W. Pleśniak ([9], pp. 21, 25), i.e.

$$\liminf_{n\to\infty} \sqrt[n]{\varrho_n(\varphi \circ \gamma, F)} = 0 ,$$

where $\varrho_n(\varphi \circ \gamma, F) := \inf\{\|\varphi \circ \gamma - P\|_F;\ P \in \mathscr{P}_n(C^M, C)\}$.
In fact it follows from (+) that for every γ the function is (n_j)-quasientire, because

$$\lim_{j\to\infty} \varrho_{n_j}(\varphi \circ \gamma, \mathscr{F})^{1/n_j} = 0 .$$

Under some additional assumption on K we shall study differentiability properties of the function φ. First we get

THEOREM 2. *If K is a polynomially convex compact subset of* C^N *such that*

$$(*) \qquad \Phi_K(x) \leqslant 1 + \varkappa\delta^{\mu}, \ x \in C^N, \ \mathrm{dist}(x, K) \leqslant \delta, \ 0 < \delta \leqslant 1,$$

where $\varkappa$, μ *are positive constants, then for every multiindex* $\alpha \in Z_+^N$ *the series* $\sum_{j=1}^{\infty} \Theta^{\sqrt{n_j}} D^{\alpha} P_j$ *is uniformly convergent on K. Here* $D^{\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_N}\right)^{\alpha_N}$.

This theorem is a direct consequence of the following Lemma 3 and of the fact that the series $\sum_{j=1}^{\infty} n_j^k \Theta^{\sqrt{n_j}}$ is convergent for every positive number k.

LEMMA 3 (Markov's inequality). *If K is a compact subset of* C^N *such that the extremal function* Φ_K *satisfies the continuity condition* (*) *then for every polynomial* $P \in \mathscr{P}_n(C^N, C)$ *and for all* $\alpha \in Z_+^N$

$$\|D^{\alpha}P\|_K \leqslant \alpha! e (\varkappa n)^{|\alpha|/\mu} \|P\|_K$$

with $\alpha! = \alpha_1! \dots \alpha_N!$ *and* $|\alpha| = \alpha_1 + \dots + \alpha_N$.

We shall see that the inequality (*) is true with $\mu = 1/2$ for all K satisfying the following

1.9. **Geometrical Condition** (GC). There exists $r > 0$ such that for every point $b \in K$ one can find a point $a \in K$ such that the convex hull of the set $\{b\} \cup B(a, r)$ is contained in K.

If K is a subset of R^N (where R^N is identified with the subset $R^N + i0$ of C^N), then it is sufficient if GC is satisfied with balls $B(a, r) = \{x \in R^N : \|x - a\| \leqslant r\}$ in R^N.

Hence and from Theorem 1, Theorem 2 and from the Whitney extension theorem we get the following

THEOREM 3. *If D is a bounded convex domain, or if D is a bounded domain with Lipschitz boundary such that* $K := \bar{D}$ *is polynomially convex, then there exists a function* $\varphi : K \to C$ *with the following properties:*

(i) φ *is* $\mathscr{C}^{\infty}$ *on* $\bar{D}$,

(ii) φ *is holomorphic in* D,

(iii) φ *has the* SNCP.

Let $D = B(0, 1)$ be the unit Euclidean ball in C^N and let (u_j) be a sequence of unit vectors dense in ∂D. It easily follows from Lemma 2 that the function f defined by

$$(\S\S) \qquad f(x) := \sum_{j=1}^{\infty} \Theta^{\sqrt{n_j}} \langle x, u_j \rangle^{n_j}, \ x \in \bar{D},$$

or by $f(x) := \sum_{j=1}^{\infty} \frac{\langle x, u_j \rangle^{n_j}}{j! \log j!}, \quad x \in \bar{D},$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbf{C}^N$, (n_j) is an increasing sequence of positive integers with $\lim n_j/n_{j+1} = 0$ and $0<\Theta<1$, is $\mathscr{C}^\infty$ on $\bar{D}$, holomorphic in D and has the SNCP. The last series was considered by the authors of [3], but they did not know if its sum had the non-continuation property for complex lines L not passing through the origin.

Observe that in the (§§) the sequence $(\Theta^{\sqrt{n_j}})_{j\geq 1}$ may be replaced by any sequence of complex number $(c_j)_{j\geq 1}$ with the following properties

(I) $\lim\limits_{j\to\infty} |c_j|^{1/n_j} = 1$,

(II) $\sum\limits_{j=1}^{\infty} n_j^k |c_j| < +\infty, \quad k = 1, 2, \ldots$

2. Proof of Lemma 1.

Let K be a compact subset of $\mathbf{C}^N$. By Theorem 1 of [12] (or by Prop. 4.11 of [15]) there exists a sequence $(Q_j)_{j\geq 1}$ of polynomials of N complex variables such that

$$k_j := \deg Q_j \leqslant k_{j+1} := Q_{j+1}$$

and

$$\Phi_K(x) = \sup_{j\geq 1} |Q_j(x)|^{1/k_j}, \; x \in \mathbf{C}^N.$$

Consider the following sequence of polynomials

(1) $$Q_1^{l_{11}}, Q_1^{l_{21}}, Q_2^{l_{22}}, Q_1^{l_{31}}, Q_2^{l_{32}}, Q_3^{l_{33}}, \ldots, Q_1^{l_{n1}}, Q_2^{l_{n2}}, \ldots, Q_n^{l_{nn}}, \ldots$$

where l_{nj} ($j = 1, \ldots, n$; $n = 1, 2, \ldots$) are positive integers. Put $n_1 = k_1 l_{11}$, $n_2 = k_1 l_{21}$, $n_3 = k_2 l_{22}, \ldots$, the numbers l_{nj} being chosen in such a way that

$$n_{j+1} \geqslant (j n_j)^2, \quad j = 1, 2, \ldots$$

Let now $(P_j)_{j\geq 1}$ denote the sequence of polynomials (1). It is obvious that the sequences $(n_j)_{j\geq 1}$ and $(P_j)_{j\geq 1}$ have the properties (i), (ii) and (iii) stated by Lemma 1.

3. Proof of Lemma 2.

1° Let F be a compact subset of D. Let G be a relatively compact subdomain of D such that $E \cup F \subset G$. Put $f_s := \sum\limits_{j=1}^{s} P_j$ and $\varepsilon_s := \|f - f_s\|_E^{1/n_s}$. Observe that

$$\|f_s\|_E \leqslant \|f\|_E + \|f - f_s\|_E = \|f\|_E + \varepsilon_s^{n_s} \leqslant M_1 = \text{const}, \; s \geqslant 1 .$$

By the Bernstein-Walsh inequality 1.1

$$|f_s(x)| \leqslant M_1 \Phi_E(x)^{n_s}, \; x \in \mathbf{C}^N, \; s \geqslant 1 .$$

Since E is nonpluripolar the function Φ_E is locally bounded. Therefore

$$\|f-f_s\|_G \leqslant M^{n_s}, \; s \geqslant 1, \; M = \text{const}.$$

Now
$$\frac{\log|f(x)-f_s(x)|^{1/n_s}-\log\varepsilon_s}{\log M-\log\varepsilon_s} \leqslant h(x, E, G), \; x \in G, \; s \geqslant 1,$$

where $h(x, E, G) := \sup\{u(x)\}$, the sup being taken over all plurisubharmonic functions u in G with $u \leqslant 0$ on E and $u \leqslant 1$ in G. Therefore

$$|f(x)-f_s(x)|^{1/n_s} \leqslant \varepsilon_s \left(\frac{M}{\varepsilon_s}\right)^{h(x,E,G)} \leqslant M_2 \varepsilon_s^{1-\vartheta}, \; x \in F, \; s \geqslant 1,$$

where M_2 is a positive constant and $\vartheta := \sup\{h(x, E, G); x \in F\}$ satisfies the inequalities $0<\vartheta<1$ (because E is nonpluripolar). It follows that for every compact subset F of D

$$\lim_{s\to\infty} \|f-f_s\|_F^{1/n_s} = 0.$$

2° Let now S be the connected component of Ω containing D. Suppose that S is not the maximal domain of existence of f. Then we can find a domain G and a holomorphic function g in G such that $S \cap G \neq \emptyset$, $G-S \neq \emptyset$ and $g = f$ on a nonempty open subset of $S \cap G$. By 1°

$$g(x) = \sum_{j=1}^{\infty} P_j(x), \; x \in G,$$

the series being locally uniformly convergent in G. It follows that $G \subset S$. This contradiction ends the proof.

4. Behaviour of the extremal function Φ_E near plane continuum E

4.1. Proposition. *If E is a compact connected subset of the complex plane $\mathbf{C}$ containing more than one point, then*

(i)
$$\Phi_E(z) \leqslant 1 + \sqrt{\frac{1}{c}\,\Phi_E(z)|z-a|}, \; z \in \mathbf{C}, \; a \in E,$$

where $c = c(E)$ is the logarithmic capacity of E;

(ii)
$$l(\Gamma_\delta) \leqslant 2\pi c \frac{(1+\delta)^3}{2+\delta}\frac{1}{\delta}, \; \delta > 0,$$

where $l(\Gamma_\delta)$ is the length of the level curve

$$\Gamma_\delta := \{z \in \mathbf{C}: \Phi_E(z) = 1+\delta\}.$$

Proof. If f is the conformal mapping of $\mathbf{C}-\hat{E}$ onto the set $\{|w|>1\}$ normalized by the condition $\lim_{z\to\infty} f(z)/z = 1/c$, then

$$\Phi_E(z) = |f(z)| \quad \text{for all } z \in \mathbf{C}-\hat{E}.$$

If a is a fixed point of $\hat{E}$ then the function

$$c/[f^{-1}(w^{-1})-a] = w+c_2w^2+\dots,\ |w|<1,$$

is univalent in the unit disk. By the Koebe distortion theorem ([4], p. 53) one gets

$$|f^{-1}(w^{-1})-a| \geqslant c(1-|w|)^2/|w|,\ |w|<1.$$

Hence by setting $w = 1/f(z)$, $z \in \boldsymbol{C}-\hat{E}$, we get

$$c(|f(z)|-1)^2 \leqslant |f(z)|\,|z-a|,\ z\in \boldsymbol{C}-\hat{E},\ a\in\hat{E},$$

which is equivalent to the inequality (i).

In order to show (ii) observe that by the inequality (21), p. 117 of [4] we get

$$\left|\frac{1}{c}\frac{d}{dw}f^{-1}(w)\right| \leqslant 1\Big/\left(1-\frac{1}{|w|^2}\right).$$

Hence

$$l(\Gamma_\delta) = \int_0^{2\pi}\left|\frac{d}{dt}f^{-1}\big((1+\delta)e^{it}\big)\right|dt \leqslant 2\pi c\frac{(1+\delta)^3}{2+\delta}\frac{1}{\delta}.$$

4.2. Corollary. *If E is a compact connected subset of $\boldsymbol{C}$ containing more than one point, then*

(a) $\Phi_E(z) \leqslant 1+\varkappa\delta^{1/2}$, $\mathrm{dist}(z, E)\leqslant\delta$, $0<\delta\leqslant 1$;

(b) $\mathrm{dist}(\Gamma_\delta, E) \geqslant \dfrac{c\delta^2}{1+\delta}$, $\delta>0$;

(c) $l(\Gamma_\delta)\leqslant 8\pi c/\delta$, $\quad 0<\delta\leqslant 1$,

where

$$\varkappa \leqslant 4\frac{(1+\|E\|)^{1/2}}{\|E\|},\qquad \|E\| := \sup\{|z-w|;\ z, w\in E\}.$$

The inequalities (b) and (c) are obvious consequences of (i) and (ii), respectively. The inequality (a) follows from (i) and from the following known inequalities

$$\Phi_E(z) \leqslant \frac{\|E\|+\mathrm{dist}(z, E)}{c}\qquad \text{(see inequality (2.16) of [13])}$$

$$4c \geqslant \|E\|\qquad \text{(see Theorem 2, p. 294 of [4])}$$

4.3. Remark. Inequality (a) was earlier obtained in [14] by a method due to F. Leja [7]. The present method based on the distortion theorem was suggested to the author by W. K. Hayman some years ago.

4.4. Corollary (Markov's inequality: see [10]). *Let K be a compact subset of $\mathbf{C}$ such that the logarithmic capacity of each connected component of K is bigger than a positive constant c_0. Then for every polynomial P of a complex variable of degree n*

$$\|P'\|_K \leqslant \frac{2e}{c_0} n^2 \|P\|_K .$$

Indeed, for a fixed point a in E let F be the connected component of E containing the point a. Then by the Cauchy formula

$$P'(a) = \frac{1}{2\pi i} \int\limits_{|z-a|=\varrho} P(z)(z-a)^{-2} dz \qquad \text{with } \varrho = \frac{c_0 \delta^2}{1+\delta}.$$

Hence

$$|P'(a)| \leqslant \|P\|_E (1+\delta)^n / \varrho = \|P\|_E (1+\delta)^{n+1} / c_0 \delta^2, \ \delta > 0 .$$

Now by setting $\delta = \frac{1}{n}$ we get the required inequality.

5. Hölder continuity property of Φ_K for a class of compact subsets K of C^N

5.1. Proposition. *Let K be a compact subset of $\mathbf{C}^N$ satisfying the following* **Condition (P):** *For each point $a = (a_1, \ldots, a_N) \in K$ there exist plane continua $E_1, \ldots, E_N$ and an affine nonsingular mapping h: $\mathbf{C}^N \ni \xi \mapsto x = h(\xi) \in \mathbf{C}^N$ of $\mathbf{C}^N$ onto $\mathbf{C}^N$ such that*

(i) $a \in h(E_1 \times \ldots \times E_N) \subset K$;

(ii) $\|E_j\| \geqslant d > 0$, $j = 1, \ldots, N$;

(iii) $\|h'\| \geqslant m > 0$,

where $\|E_j\|$ is the diameter of E_j and the constants d and m do not depend on the point a; the expression $\|h'\|$ denotes the norm of the Fréchet derivative of h (the derivative h' is a linear mapping of $\mathbf{C}^N$ onto $\mathbf{C}^N$ and does not depend on the point at which it is computed).

Then there exists a positive constant $\varkappa$ such that

$$\Phi_K(x) \leqslant 1 + \varkappa \delta^{1/2}, \ \mathrm{dist}(x, K) \leqslant \delta, \qquad 0 < \delta \leqslant 1 .$$

Proof. Given a point $a \in K$ let $\alpha = (\alpha_1, \ldots, \alpha_N)$ be a point of $\mathbf{C}^N$ with $a = h(\alpha)$. It is obvious that

$$\Phi_{h(E_1 \times \ldots \times E_N)}(x) = \Phi_{E_1 \times \ldots \times E_N}(h^{-1}(x)) \text{ in } \mathbf{C}^N.$$

Therefore by Properties 1.4, 1.6 and by Corollary 4.2

$$\Phi_K(x) \leqslant \Phi_{E_1 \times \ldots \times E_N}(h^{-1}(x))$$

and

$$\Phi_{E_1 \times \ldots \times E_N}(\xi) \leqslant \max_{1 \leqslant j \leqslant N} \{1 + \varkappa_j |\xi_j - \alpha_j|^{1/2}\} \leqslant 1 + \varkappa_0 |\xi - \alpha|^{1/2},$$

where $\varkappa_j \leqslant \frac{4}{d}\sqrt{1+||E||}$, $j = 0, \ldots, N$. Thus

$$\Phi_K(x) \leqslant 1+\varkappa_0|h^{-1}(x)-h^{-1}(a)|^{1/2} \leqslant 1+\frac{\varkappa_0}{||h'||}|x-a|^{1/2}$$

for all $x \in \mathbf{C}^N$, $a \in K$ with $|x-a| \leqslant \delta \leqslant 1$. We see that the required inequality is true with $\varkappa = \varkappa_0/m$ and $\varkappa \leqslant 4\sqrt{1+||E||}/(md)$.

5.2. Corollary. *Let D be a bounded domain in $\mathbf{R}^N$ (resp. in $\mathbf{C}^N$) with Lipschitz boundary. Then $K = \bar{D}$ satisfies the condition* (P) *of Proposition* 5.1.

Indeed, as observed in [1], pp. 166-167, the set $K = \bar{D}$ satisfies the following

Property (H_2). There exists a nonempty parallelepiped π_0 such that each point x of K is a vertex of a parallelepiped π_x congruent with π_0 (with respect to orthogonal transformations and translations) and verifying $\pi_x \subset K$.

It is obvious that $(H_2) \Rightarrow (P)$.

5.3. Proposition. *Let K be a polynomially convex compact subset of $\mathbf{C}^N$ satisfying the Geometrical Condition* 1.9. *Then there exists a positive constant $\varkappa$ such that*

$$\Phi_K(x) \leqslant 1+\varkappa\delta^{1/2}, \ \operatorname{dist}(x, K) \leqslant \delta, \quad 0<\delta\leqslant 1 .$$

Proof. It is enough to show that K satisfies the Condition (P). First we shall consider the case when K is a subset of $\mathbf{R}^N$ satisfying the Geometrical Condition 1.9. Given a fixed point b of K let a be another point of K such that the convex hull of $\{b\} \cup B(a, r)$ is contained in K. Without loss of generality we may assume that $||b-a|| \geqslant r$. Let $\mathbf{c}^2, \ldots, \mathbf{c}^N$ be points in the ball $B = B(a, r)$ such that $||\mathbf{c}^j-a|| = r$ $(j = 2, \ldots, N)$ and the vectors $b-a$, $\mathbf{c}^2, \ldots, \mathbf{c}^N$ are orthogonal to each other. The affine mapping

$$h(\xi) := a+\frac{1}{N}[\xi_1(b-a)+\xi_2(\mathbf{c}^2-a)+\ldots+\xi_N(\mathbf{c}^N-a)], \ \xi = (\xi_1, \ldots, \xi_N) \in \mathbf{C}^N,$$

transforms the cube I^N with $I = [0, 1]$ onto a subset of K containing the points $a, b, \mathbf{c}^2, \ldots$ $\ldots, \mathbf{c}^N$. Moreover

$$\begin{aligned}\det h' &= N^{-N}(b-a)\wedge(\mathbf{c}^2-a)\wedge \ldots \wedge(\mathbf{c}^N-a) = \\ &= N^{-N}||b-a||\,||\mathbf{c}^2-a|| \ldots ||\mathbf{c}^N-a|| \geqslant (r/N)^N.\end{aligned}$$

Observe that $h'(\xi) = h(\xi)-a$ and $(h^{-1})' = (h')^{-1}$. Therefore

$$||(h^{-1})'|| \leqslant N! \max_{1\leqslant i,j\leqslant N} |\mathbf{c}_{ij}|,$$