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Multiple Differentiability of Solutions with Respect to Parameter

by Teresa WINIARSKA

Abstract. In this paper we will prove some theorems on continuity and k-times differentiability with respect to parameter λ of the solution of a non-linear equation

$$\frac{\partial u}{\partial t}+G(t, \lambda, u(t, \lambda))=f(t, \lambda), \qquad t\in(0, T], \qquad \lambda\in R$$

with the initial condition

$$u(0, \lambda) = \varphi(\lambda), \qquad \lambda\in R.$$

Let X be a Banach space and let $S = [0, T]$, $T > 0$ be a closed interval in R.

Let $\{G(t, \lambda, \cdot)\}$ be a family of non-linear (or linear) operators $G(t, \lambda, \cdot)\colon X\to X$, $t\in S$, $\lambda\in R$ and let $f\colon S\times R\to X$, $u\colon S\times R\to X$ be two mappings.

We shall consider the equation

$$\frac{\partial u}{\partial t}+G(t, \lambda, u(t, \lambda))=f(t, \lambda), t\in(0, T], \qquad \lambda\in R \tag{1}$$

with the initial condition

$$u(0, \lambda) = \varphi(\lambda), \qquad \lambda\in R, \tag{2}$$

where the mapping $\varphi\colon R\to X$ is given.

We now state three known results that will be useful to us.

LEMMA 1 (*Gronwall*, see e.g. [1] Lemma 1.3). *Let $f\colon S\to R$ be continuous and suppose that $g\colon S\to R$ is non-decreasing. If for every $t\in S$*

$$f(t)\leqslant g(t)+c\int_0^t f(s)ds$$

then

$$f(t)\leqslant e^{ct}g(t) \qquad \text{for } t\in S.$$

Let us now state an inconsiderable generalization of a known result (cf. [1], Lemma 1.3).

LEMMA 2. *Let* $\mathscr{F}$ *be a topological space and let a mapping* $H\colon \mathscr{F}\times X\to X$ *be such that*

1) *for every* $x\in X$ *the mapping* $\mathscr{F}\ni\tau\to H(\tau,x)$ *is continuous,*

2) *for every* $\tau\in\mathscr{F}$ *there exists a constant* $L_\tau>0$ *such that*

$$\|H(\tau,x)-H(\tau,y)\|\leqslant L_\tau\|x-y\| \quad \textit{for every } (x,y)\in X\times X,$$

3) *the function* $\mathscr{F}\ni\tau\to L_\tau\in R_+$ *is locally bounded.*

If a mapping $v\colon \mathscr{F}\to X$ *is continuous, then the mapping*

$$\Phi\colon \mathscr{F}\ni\tau\to H(\tau,v(\tau))\in X$$

is continuous too.

Proof. Let us fix $\varepsilon>0$ and $\tau_0\in\mathscr{F}$. Then there exist $L>0$ and a neighborhood U of τ_0 such that $L_\tau\leqslant L$, $\|v(\tau)-v(\tau_0)\|<\frac{\varepsilon}{2L}$ and $\|H(\tau,v(\tau_0))-H(\tau_0,v(\tau_0))\|<\frac{\varepsilon}{2}$ for every $\tau\in U$. If $\tau\in U$, then

$$\begin{aligned}\|\Phi(\tau)-\Phi(\tau_0)\| &= \|H(\tau,v(\tau))-H(\tau_0,v(\tau_0))\|\\ &\leqslant \|H(\tau,v(\tau))-H(\tau,v(\tau_0))\|+\|H(\tau,v(\tau_0))-H(\tau_0,v(\tau_0))\|\\ &\leqslant L\|v(\tau)-v(\tau_0)\|+\frac{\varepsilon}{2}<L\frac{\varepsilon}{2L}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Theorem 1 ([1], p. 192). *Let for every* $(\lambda,x)\in R\times X$ *the function* $S\ni t\to G(t,\lambda,x)$ *be continuous. Suppose that for every* $\lambda\in R$ *there exists a constant* $L_\lambda>0$ *such that for every* $(t,x,y)\in S\times X\times X$

$$\|G(t,\lambda,x)-G(t,\lambda,y)\|\leqslant L_\lambda\|x-y\|. \tag{3}$$

Let $f\colon S\times R\to X$ *be continuous. Then, for every* $\lambda\in R$, *the problem* (1)—(2) *has exactly one solution* u *satisfying the formula*

$$u(t,\lambda)=\varphi(\lambda)-\int_0^t\big(G(s,\lambda,u(s,\lambda))-f(s,\lambda)\big)ds. \tag{4}$$

Moreover, the mapping

$$S\ni t\to u(t,\lambda)\in X$$

is of class C^1.

Now, we shall prove a theorem on the continuity of the solution of the problem (1)—(2) with respect to both t and parameter λ.

THEOREM 2. *Suppose that the following five conditions hold:*

1) *for every* $x\in X$ *the mapping* $S\times R\ni(t,\lambda)\to G(t,\lambda,x)\in X$ *is continuous,*

2) *for every* $\lambda\in R$ *there exists a constant* L_λ *such that* $\|G(t,\lambda,x)-G(t,\lambda,y)\|\leqslant L_\lambda\|x-y\|$ *for* $(t,x,y)\in S\times X\times X$,

3) *the function* $R\in\lambda\to L_\lambda$ *is locally bounded,*

4) *the function* $\varphi\colon R\ni\lambda\to\varphi(\lambda)\in X$ *is continuous,*

5) *the function* $f\colon S\times R\ni(t,\lambda)\to f(t,\lambda)\in X$ *is continuous. Then for every* $\lambda\in R$ *there*

exists exactly one solution $u(\cdot, \lambda)$ of the problem (1)—(2). *Moreover, this solution is continuous in $S \times R$.*

Proof. Let us fix $\lambda_0 \in R$ and $t_0 \in S$. By letting $\lambda = \lambda_0$ in (4) we obtain

$$u(t, \lambda_0) = \varphi(\lambda_0) - \int_0^t \big(G(s, \lambda_0, u(s, \lambda_0)) - f(s, \lambda_0)\big) ds .$$

Then

$$\|u(t, \lambda) - u(t, \lambda_0)\| = \|\varphi(\lambda) - \varphi(\lambda_0) - \int_0^t \big[G(s, \lambda, u(s, \lambda)) - G(s, \lambda_0, u(s, \lambda_0))\big] ds$$

$$+ \int_0^t [f(s, \lambda) - f(s, \lambda_0)] ds \leqslant \|\varphi(\lambda) - \varphi(\lambda_0)\|$$

$$+ \int_0^t \|f(s, \lambda) - f(s, \lambda_0)\| ds + \int_0^t \|G(s, \lambda, u(s, \lambda)) - G(s, \lambda, u(s, \lambda_0))\| ds$$

$$+ \int_0^t \|G(s, \lambda, u(s, \lambda_0)) - G(s, \lambda_0, u(s, \lambda_0))\| ds$$

It follows from assumptions 2) and 3) that there exists $\delta_1 > 0$ and $L > 0$ such that the conditions $|\lambda - \lambda_0| < \delta_1$, $t \in S$ imply that

$$\|u(t, \lambda) - u(t, \lambda_0)\| \leqslant \|\varphi(\lambda) - \varphi(\lambda_0)\| + \int_0^t \|f(s, \lambda) - f(s, \lambda_0)\| ds$$

$$+ \int_0^t \|G(s, \lambda, u(s, \lambda_0)) - G(s, \lambda_0, u(s, \lambda_0))\| ds + L \int_0^t \|u(s, \lambda) - u(s, \lambda_0)\| ds$$

If we set $|\lambda - \lambda_0| < \delta_1$, $t \in S$, then it follows from Lemma 1 that

$$\|u(t, \lambda) - u(t, \lambda_0)\| \leqslant K\Big(\|\varphi(\lambda) - \varphi(\lambda_0)\| + \int_0^t \|f(s, \lambda) - f(s, \lambda_0)\| ds$$

$$+ \int_0^t \|G(s, \lambda, u(s, \lambda_0)) - G(s, \lambda_0, u(s, \lambda_0))\| ds ,$$

where $K = \exp(LT)$.

Let us take an arbitrary $\varepsilon > 0$. Since φ is continuous, there exists $\delta_2 < \delta_1$ such that the condition $|\lambda - \lambda_0| < \delta_2$ implies that $\|\varphi(\lambda) - \varphi(\lambda_0)\| < \frac{\varepsilon}{6}$.

Let us observe that

$$\int_0^t \|f(s, \lambda) - f(s, \lambda_0)\| ds \leqslant \int_0^T \|f(s, \lambda) - f(s, \lambda_0)\| ds ,$$

and that f is uniformly continuous on the set $S \times \{\lambda : |\lambda - \lambda_0| \leqslant \delta_2\}$. It follows from the above that there exists $\delta_3 < \delta_2$ such that

$$\|f(s, \lambda) - f(s, \lambda)_0\| < \frac{\varepsilon}{6T} \quad \text{for } (s, t) \in S \times \{\lambda : |\lambda - \lambda_0| < \delta_3\} .$$

Hence

$$\int_0^t \|f(s,\lambda)-f(s,\lambda_0)\|\,ds<\frac{\varepsilon}{6} \quad \text{holds with } |\lambda-\lambda_0|<\delta_3 .$$

Similarly, applying Lemma 2, we conclude that there exists $\delta<\delta_3$ such that

$$\int_0^t \|G(s,\lambda,u(s,\lambda))-G(s,\lambda_0,u(s,\lambda_0))\|\,ds<\frac{\varepsilon}{6} \quad \text{holds with } |\lambda-\lambda_0|<\delta$$

Finally, if $t\in S$ and $|\lambda-\lambda_0|<\delta$, then $\|u(t,\lambda)-u(t,\lambda_0)\|<\frac{\varepsilon}{2}$. Theorem 1 states that there exists $\mu>0$ such that

$$\|u(t,\lambda_0)-u(t_0,\lambda_0)\|<\frac{\varepsilon}{2} \quad \text{holds with } |t-t_0|<\mu .$$

Let us now summarize. If we set $U=\{(t,\lambda)\in S\times R\colon\ |t-t_0|<\mu,\ |\lambda-\lambda_0|<\delta\}$ then for $(t,\lambda)\in U$ we have

$$\|u(t,\lambda)-u(t_0,\lambda_0)\|\leqslant\|u(t,\lambda)-u(t,\lambda_0)\|+\|u(t,\lambda_0)-u(t_0,\lambda_0)\|<\varepsilon .$$

Since (t_0,λ_0) is an arbitrary point of $S\times R$, the solution u is continuous in $S\times R$.

Theorem 3. *Let us suppose that*

1) *all assumptions of Theorem 2 are satisfied,*
2) *for every* $t\in S$ *the mapping* $R\times X\ni(\lambda,x)\to G(t,\lambda,x)$ *is differentiable*
3) $\frac{\partial G}{\partial x}$, $\frac{\partial G}{\partial \lambda}$ *are continuous in* $S\times R\times X$,
4) *for every* $t\in S$ *the mapping* $R\ni\lambda\to f(t,\lambda)$ *is differentiable,*
5) $\frac{\partial f}{\partial \lambda}$ *is continuous in* $S\times R$,
6) φ *is differentiable in* R.

Then the solution u *of the problem* (1)—(2) *is differentiable with respect to* λ.

Proof. Let us take $\lambda\neq\lambda_0$. Then

$$\frac{\partial u}{\partial t}(t,\lambda)-\frac{\partial u}{\partial t}(t,\lambda_0)+\big(G(t,\lambda,u(t,\lambda))-G(t,\lambda_0,u(t,\lambda))\big)$$
$$+\big(G(t,\lambda_0,u(t,\lambda))-G(t,\lambda_0,u(t,\lambda_0))\big)=f(t,\lambda)-f(t,\lambda_0)$$

and $u(0,\lambda)-u(0,\lambda_0)=\varphi(\lambda)-\varphi(\lambda_0)$.

Applying the Peano formula to differences in the brackets and to the right-hand sides we obtain

$$\begin{aligned}(5)\qquad & \frac{\partial u}{\partial t}(t, \lambda) - \frac{\partial u}{\partial t}(t, \lambda_0) + \frac{\partial G}{\partial \lambda}(t, \lambda_0, u(t, \lambda))(\lambda - \lambda_0) + \psi_t(\lambda - \lambda_0) \\ & + \frac{\partial G}{\partial x}(t, \lambda_0, u(t, \lambda_0))(u(t, \lambda) - u(t, \lambda_0)) + \eta_t(u(t, \lambda) - u(t, \lambda_0)) \\ & = \frac{\partial f}{\partial \lambda}(t, \lambda_0)(\lambda - \lambda_0) + \zeta_t(\lambda - \lambda_0)\end{aligned}$$

$$(6)\qquad u(0, \lambda) - u(0, \lambda_0) = \varphi'(\lambda_0)(\lambda - \lambda_0) + \varkappa(\lambda - \lambda_0)$$

Dividing both sides of (5), (6) by $(\lambda - \lambda_0)$ we have

$$\begin{aligned}& \frac{\partial}{\partial t}\left[\frac{u(t, \lambda) - u(t, \lambda_0)}{\lambda - \lambda_0}\right] + \frac{\partial G}{\partial x}(t, \lambda_0, u(t, \lambda_0))\frac{u(t, \lambda) - u(t, \lambda_0)}{\lambda - \lambda_0} \\ & + \frac{\eta_t(u(t, \lambda) - u(t, \lambda))}{\|u(t, \lambda) - u(t, \lambda_0)\|}\,\frac{\|u(t, \lambda) - u(t, \lambda_0)\|}{\lambda - \lambda_0} = \frac{\partial f}{\partial \lambda}(t, \lambda_0) \\ & - \frac{\partial G}{\partial \lambda}(t, \lambda_0, u(t, \lambda)) - \frac{\psi_t(\lambda - \lambda_0)}{\lambda - \lambda_0} + \frac{\zeta_t(\lambda - \lambda_0)}{\lambda - \lambda_0},\end{aligned}$$

$$\frac{u(0, \lambda) - u(0, \lambda_0)}{\lambda - \lambda_0} = \varphi'(\lambda_0) + \frac{\varkappa(\lambda - \lambda_0)}{\lambda - \lambda_0}$$

If we set

$$F(t, \lambda, \lambda_0) := \begin{cases} \dfrac{\partial f}{\partial \lambda}(t, \lambda_0) - \dfrac{\partial G}{\partial \lambda}(t, \lambda_0, u(t, \lambda)) - \dfrac{\psi_t(\lambda - \lambda_0)}{\lambda - \lambda_0} + \dfrac{\zeta_t(\lambda - \lambda_0)}{\lambda - \lambda_0} & \text{for } \lambda \neq \lambda_0 \\ \dfrac{\partial f}{\partial \lambda}(t, \lambda_0) - \dfrac{\partial G}{\partial \lambda}(t, \lambda_0, u(t, \lambda_0)) & \text{for } \lambda = \lambda_0 \end{cases}$$

$$\Phi(\lambda, \lambda_0) := \begin{cases} \varphi'(\lambda_0) + \dfrac{\varkappa(\lambda - \lambda_0)}{\lambda - \lambda_0} & \text{for } \lambda \neq \lambda_0 \\ \varphi'(\lambda_0) & \text{for } \lambda = \lambda_0 \end{cases}$$

then we have

$$\begin{aligned}(7)\qquad & \frac{\partial}{\partial t}\left[\frac{u(t, \lambda) - u(t, \lambda_0)}{\lambda - \lambda_0}\right] + \frac{\partial G}{\partial x}(t, \lambda_0, u(t, \lambda_0))\frac{u(t, \lambda) - u(t, \lambda_0)}{\lambda - \lambda_0} \\ & \qquad + \frac{\eta_t(u(t, \lambda) - u(t, \lambda_0)}{\|u(t, \lambda) - u(t, \lambda_0)\|}\,\frac{\|u(t, \lambda) - u(t, \lambda_0)\|}{\lambda - \lambda_0} = F(t, \lambda, \lambda_0)\end{aligned}$$

$$(8)\qquad \frac{u(0, \lambda) - u(0, \lambda_0)}{\lambda - \lambda_0} = \Phi(\lambda, \lambda_0)$$

Let us observe that

$$\frac{\eta_t(u(t,\lambda)-u(t,\lambda_0))}{\|u(t,\lambda)-u(t,\lambda_0)\|}\,\frac{\|u(t,\lambda)-u(t,\lambda_0)\|}{\lambda-\lambda_0}$$

$$= \operatorname{sgn}(\lambda-\lambda_0)\,\frac{\eta_t(u(t,\lambda)-u(t,\lambda_0))}{\|u(t,\lambda)-u(t,\lambda_0)\|}\left\|\frac{u(t,\lambda)-u(t,\lambda_0)}{\lambda-\lambda_0}\right\|$$

Letting $\omega(t,\lambda,\lambda_0) = \operatorname{sgn}(\lambda-\lambda_0)\,\dfrac{\eta_t(u(t,\lambda)-u(t,\lambda_0))}{\|u(t,\lambda)-u(t,\lambda_0)\|}$

we have

$$\text{(9)}\qquad \frac{\eta_t(u(t,\lambda)-u(t,\lambda_0))}{\|u(t,\lambda)-u(t,\lambda_0)\|}\,\frac{\|u(t,\lambda)-u(t,\lambda_0)\|}{\lambda-\lambda_0} = \omega(t,\lambda,\lambda_0)\left\|\frac{u(t,\lambda)-u(t,\lambda_0)}{\lambda-\lambda_0}\right\|$$

By letting (9) in (7) we obtain

$$\text{(10)}\qquad \frac{\partial}{\partial t}\left[\frac{u(t,\lambda)-u(t,\lambda_0)}{\lambda-\lambda_0}\right]+\frac{\partial G}{\partial x}(t,\lambda_0,u(t,\lambda_0))\,\frac{u(t,\lambda)-u(t,\lambda_0)}{\lambda-\lambda_0}$$

$$+\omega(t,\lambda,\lambda_0)\left\|\frac{u(t,\lambda)-u(t,\lambda_0)}{\lambda-\lambda_0}\right\| = F(t,\lambda,\lambda_0)$$

Let us write

$$\vartheta(t,\lambda,\lambda_0) = \frac{u(t,\lambda)-u(t,\lambda_0)}{\lambda-\lambda_0}$$

and

$$A(t,\lambda,\lambda_0,w) := \begin{cases} \dfrac{\partial G}{\partial x}(t,\lambda_0,u(t,\lambda_0))w+\omega(t,\lambda,\lambda_0)\|w\| & \text{for } \lambda \neq \lambda_0 \\[2ex] \dfrac{\partial G}{\partial x}(t,\lambda_0,u(t,\lambda_0))w & \text{for } \lambda = \lambda_0 \end{cases}$$

for any $w \in X$.

Now, equalities (10) and (8) have the forms

$$\text{(11)}\qquad \frac{\partial\vartheta}{\partial t}+A(t,\lambda,\lambda_0,\vartheta(t,\lambda,\lambda_0)) = F(t,\lambda,\lambda_0)$$

$$\text{(12)}\qquad \vartheta(0,\lambda) = \Phi(\lambda,\lambda_0)$$

Straightforward computation yields that the operator $A(t,\lambda,\lambda_0,\cdot)\colon X\to X$ satisfies condition (3). Indeed, for $\lambda \neq \lambda_0$ we have

$$\|A(t,\lambda,\lambda_0,\vartheta_1)-A(t,\lambda,\lambda_0,\vartheta_2)\| = \left\|\frac{\partial G}{\partial x}(t,\lambda_0,u(t,\lambda_0))\vartheta_1+\omega(t,\lambda,\lambda_0)\|\vartheta_1\|\right.$$

$$\left.-\frac{\partial G}{\partial x}(t,\lambda_0,u(t,\lambda_0))\vartheta_2-\omega(t,\lambda,\lambda_0)\|\vartheta_2\|\right\|$$

$$= \left\| \frac{\partial G}{\partial x}(t, \lambda_0, u(t, \lambda_0))(\vartheta_1 - \vartheta_2) + \omega(t, \lambda, \lambda_0)(\|\vartheta_1\| - \|\vartheta_2\|) \right\|$$

$$\leqslant \left\| \frac{\partial G}{\partial x}(t, \lambda_0, u(t, \lambda_0)) \right\| \|\vartheta_1 - \vartheta_2\| + \|\omega(t, \lambda, \lambda_0)\| \left| \|\vartheta_1\| - \|\vartheta_2\| \right|$$

$$\leqslant \left(\left\| \frac{\partial G}{\partial x}(t, \lambda_0, u(t, \lambda_0)) \right\| + \|\omega(t, \lambda, \lambda_0)\| \right) \|\vartheta_1 - \vartheta_2\| = M(t, \lambda, \lambda_0) \|\vartheta_1 - \vartheta_2\|$$

where $M\colon S \times R \ni (t, \lambda) \to M(t, \lambda, \lambda_0) \in R_+$ is continuous. Therefore, for every $\delta > 0$ there exists $M > 0$ such that the condition $(t, \lambda) \in S \times [\lambda_0 - \delta, \lambda_0 + \delta]$ implies that $M(t, \lambda, \lambda_0) < M$. Hence

$\|A(t, \lambda, \lambda_0, \vartheta_1) - A(t, \lambda, \lambda_0, \vartheta_2)\| \leqslant M\|\vartheta_1 - \vartheta_2\|$ for $(t, \lambda) \in S \times [\lambda_0 - \delta, \lambda_0 + \delta]$, $\lambda \neq \lambda_0$

If $\lambda = \lambda_0$ then

$$A(t, \lambda, \lambda_0, w) = \frac{\partial G}{\partial x}(t, \lambda_0, u(t, \lambda_0)) w$$

is a bounded linear operator. Therefore it satisfies condition (3).

We see at once that the problem (11)—(12) has the form (1)—(2) with $G = A$. Moreover, operator A, the right-hand side of (11) and the initial condition (12) satisfy all the assumptions of Theorem 2. Since ϑ is the solution of the problem (11)—(12), it follows from Theorem 2 that ϑ is continuous in $S \times R$. By letting $\lambda = \lambda_0$ we obtain

$$\vartheta(t, \lambda_0) = \lim_{\lambda \to \lambda_0} \frac{u(t, \lambda) - u(t, \lambda_0)}{\lambda - \lambda_0} = \frac{\partial u}{\partial \lambda}(t, \lambda_0).$$

Hence u is differentiable with respect to λ. Passing in the problem (11)—(12) to the limit for $\lambda \to \lambda_0$, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial \lambda}\right) + \frac{\partial G}{\partial x}(t, \lambda_0, u(t, \lambda_0)) \frac{\partial u}{\partial \lambda} = \frac{\partial f}{\partial \lambda}(t, \lambda_0) - \frac{\partial G}{\partial \lambda}(t, \lambda_0, u(t, \lambda_0))$$

and

$$\frac{\partial u}{\partial \lambda}(0, \lambda_0) = \varphi'(\lambda_0).$$

Since $\lambda_0 \in R$ is arbitrary, we conclude that for every $\lambda \in R$, $t \in (0, T]$ the partial derivative $\frac{\partial u}{\partial \lambda}$ of the solution u of problem (1)—(2) satisfies the following equality

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial \lambda}\right) + \frac{\partial G}{\partial x}(t, \lambda, u(t, \lambda)) \frac{\partial u}{\partial \lambda} = \frac{\partial f}{\partial \lambda}(t, \lambda) - \frac{\partial G}{\partial \lambda}(t, \lambda, u(t, \lambda)) \text{ in } (0, T] \times R$$

and the initial condition

$$\frac{\partial u}{\partial \lambda}(0, \lambda) = \varphi'(\lambda) \quad \text{for } \lambda \in R.$$

Theorem 4. *Let all the assumptions of Theorem 3 be satisfied. If $\varphi\colon R \to X$ is of class C^1, then the derivative $\frac{\partial u}{\partial \lambda}$ of the solution u of problem* (1)—(2) *is the unique solution of the problem*

$$\frac{\partial \vartheta}{\partial t}+\frac{\partial G}{\partial x}(t,\lambda,u(t,\lambda))\vartheta=\frac{\partial f}{\partial \lambda}(t,\lambda)-\frac{\partial G}{\partial \lambda}(t,\lambda,u(t,\lambda)), \tag{13}$$

$$\vartheta(0,\lambda)=\varphi'(\lambda), \tag{14}$$

with $(t,\lambda)\in S\times R$. Moreover, the mapping $S\times R \ni (t,\lambda)\to \frac{\partial u}{\partial \lambda}(t,\lambda)\in X$ is continuous.

Proof. By Theorem 3 there exists the derivative $\frac{\partial u}{\partial \lambda}$ of the solution of the problem (1)—(2) and it is a solution of (13). Straightforward computation using the fact that φ is of class C^1, and (2) yields that $\vartheta=\frac{\partial u}{\partial \lambda}$ satisfies the initial condition (14). Assumptions of Theorem 4 imply that problem (13)—(14) satisfies all the assumptions of Theorem 2. Therefore, by Theorem 2, the solution $\frac{\partial u}{\partial \lambda}$ is continuous.

Let us observe that the above theorems enable us to state new theorems on the higher order regularity with respect to parameter λ of the solution of problem (1)—(2). For example, if we would like to get a theorem on existence of the second order derivative $\frac{\partial^2 u}{\partial \lambda^2}$ of the solution u of problem (1)—(2), it suffices to assume the following four conditions:

1) all the assumptions of Theorem 3,

2) the existence and continuity in $S\times R\times X$ of mappings $\frac{\partial^2 G}{\partial x^2}, \frac{\partial^2 G}{\partial \lambda^2}, \frac{\partial^2 G}{\partial \lambda \partial x}$,

3) the continuity of $\frac{\partial^2 f}{\partial \lambda^2}$ in $S\times R$,

4) the continuity of φ'' in R.

Let us end with one example.

Example. Let $X=R^n$, $n\geqslant 1$ and let mappings $F=(F_1,\ldots,F_n)\colon S\times R\times X\to X$, $f=(f_1,\ldots,f_n)\colon S\times R\to X$, $\varphi=(\varphi_1,\ldots,\varphi_n)\colon R\to X$ be given. If we set $G(t,\lambda,x)=f(t,\lambda)-F(t,\lambda,x)$, (1)—(2) becomes the system, with parameter, ordinary differential equations

$$\frac{du_i}{dt}=F_i(t,\lambda,u_1,\ldots,u_n), \qquad i=1,\ldots,n \tag{15}$$

with the initial condition

$$u_i(0, \lambda) = \varphi_i(\lambda), \quad i = 1, \ldots, n. \tag{16}$$

Theorems 1, 2, 3, 4 are the existence, uniqueness, continuity and differentiability of solution of the linear problem (15)—(16) with respect to parameter λ.

Reference

[1] H. Gajewski, K. Groger, K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie-Verlag Berlin 1974.

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