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An Asymptotic Distribution of Eigenvalues and Eigenfunctions of the Operator Δ^{2r}, $r \geqslant 1$

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Introduction. This note is the first part of the more extensive study about the inverse Sturm-Liouville problem. One of the methods used in the proofs presented in the papers [1], [2], [3], [9] is based on some asymptotic properties of eigenvalues and eigenfunctions of that problem. In these papers, concerning differential equations of the second order, the extremum principle is of a great importance.

The purpose of this note (as a whole) is to extend results of the papers [3], [2] on a certain class of partial differential equations of higher order. The way of the proof used in [3], [2] descending from the extremum principle, can not be followed in this case. For this reason the method applied in this note is essentially different from the approach used in earlier papers on that subject.

§ 1. Sequence of eigenvalues and eigenfunctions of the operator Δ^{2^r}, $r \geqslant 1$

Let D be a bounded domain in R^m and let ∂D be its boundary. We define usual Sobolev spaces:

$$W_2^{2r}(D) := \{u;\ u \in L^2(D),\ D^\alpha u \in L^2(D),\ |\alpha| \leqslant 2r\}$$

$$W_2^{2r}(bc) := \left\{u;\ u \in W_2^{2r}(D) \frac{du}{dn} - \sigma u|_{\partial D} = 0, \ldots, \frac{d(\Delta^r u)}{dn} - \sigma u|_{\partial D} = 0,\ \sigma \in C^{2r-1}(\partial D)\right\}$$

The operator $-\Delta$ defined for $u \in W_2^2(bc)$, for $\sigma(x)|_{\partial D} \leqslant 0$ is symmetric, non — negative and can be uniquely extended to a self-adjoint operator. For the following Problems

$$\Delta u + \lambda u = 0 \quad u \in W_2^2(bc) \tag{1}$$

$$\Delta^{2r} u - \lambda^{2r} u = 0 \quad u \in W_2^{2r}(bc) \tag{2}$$

known properties of symmetric operators justify the following.

THEOREM 1. *If $\{\varphi_n\}_{n \in N}$ is a complete sequence of eigenfunctions corresponding to eigenvalues $\{\lambda_n\}_{n \in N}$ for the Problem* (1), *then $\{\varphi_n\}$ is a complete sequence of eigenfunctions corresponding to eigenvalues $\{\lambda_n^{2r}\}_{n \in N}$ for the Problem* (2).

§ 2. Some properties of number series. In this section we are going to prove some properties of number series. They are needed for a formula for an asymptotic distribution of eigenvalues and eigenfunctions.

Lemma 1. *If sequences of real numbers* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *fulfil conditions*

1° $A_n = 0(n^{1-2\alpha})$, where $A_n := \sum_{k=1}^{n} a_k$

2° $0 < b_n \leqslant b_{n+1}$, $b_n = O(n^{1+\alpha})$

3° $0 < c_n \leqslant c_{n+1}$, $c_n = O(n^{2\alpha})$

4° $\dfrac{c_n}{b_n} \geqslant \dfrac{c_{n+1}}{b_{n+1}}$,

where $0 < \alpha < \frac{1}{2}$, *then the series*

$$\sum_{n=1}^{\infty} \frac{a_n c_n}{b_n} \tag{3}$$

converges.

Proof. Note that the series $\sum_{n=1}^{\infty} \dfrac{1}{b_n}$ converges. Integrating this series by parts we get

$$\sum_{k=1}^{n} \frac{1}{b_k} = \frac{n}{b_n} + \sum_{k=1}^{n-1} k\left(\frac{1}{b_k} - \frac{1}{b_{k+1}}\right) \tag{4}$$

Equality (4) holds for each $n \in N$. Assumption 2° yields $\lim_{n\to\infty} \dfrac{n}{b_n} = 0$, then (4) implies that the series

$$\sum_{n=1}^{\infty} n\left(\frac{1}{b_n} - \frac{1}{b_{n+1}}\right) \tag{5}$$

has non-negative elements and converges. Integrating by parts series (3) too, we obtain

$$\sum_{k=1}^{n} \frac{a_k c_k}{b_k} = A_n \frac{c_n}{b_n} + \sum_{k=1}^{n-1} A_k\left(\frac{c_k}{b_k} - \frac{c_{k+1}}{b_{k+1}}\right) \tag{6}$$

Note that $\lim_{n\uparrow\infty} A_n \dfrac{c_n}{b_n} = 0$
then the convergence of the series

$$\sum_{n=1}^{\infty} A_n\left(\frac{c_n}{b_n} - \frac{c_{n+1}}{b_{n+1}}\right) \tag{7}$$

will complete the proof of Lemma 1. To prove that convergence let us observe that

$$\left|A_n\left(\frac{c_n}{b_n}-\frac{c_{n+1}}{b_{n+1}}\right)\right| = |A_n|\left(\frac{c_n}{b_n}-\frac{c_{n+1}}{b_{n+1}}\right)\leqslant|A_n|\left(\frac{1}{b_n}-\frac{1}{b_{n+1}}\right)c_n\leqslant C\cdot n\left(\frac{1}{b_n}-\frac{1}{b_{n+1}}\right)$$

for n big enough, where C is a positive constant.

This inequality means that the elements of the series

$$C\cdot\sum_{n=1}^{\infty} n\left(\frac{1}{b_n}-\frac{1}{b_{n+1}}\right)$$

are greater than absolute values of the elements of (7). It converges because of (5), then (7) converges as well. Thus Lemma 1 is proved.

We are going to use Lemma 1 in the proof of the convergence of the series

$$\sum_{n=1}^{\infty}\frac{\varphi_n^2(x_0)-\varphi_n^2(y_0)}{\lambda_n^{m/2-1/4}} \tag{8}$$

where $x_0, y_0 \in D$ are fixed points, $\{\varphi_n\}_{n\in N}$ a sequence of eigenfunctions, $\{\lambda_n\}_{n\in N}$ a sequence of eigenvalues of Problem (1), and D is a bounded domain in R^m, $m\geqslant 2$.

By the asymptotic equality [[8], p. 240, (18.7.4)] we get

$$\sum_{k=1}^{n}\varphi_k^2(x) = \frac{\lambda_n^{m/2}}{2^m\pi^{m/2}\Gamma\left(\frac{m}{2}+1\right)}+O(\lambda_n^{m/2-1/2}) \text{ for } n\to\infty . \tag{9}$$

(9) yields

$$\sum_{k=1}^{n}[\varphi_k^2(x_0)-\varphi_k^2(y_0)] = O(\lambda_n^{m/2-1/2}) \quad \text{for } n\to\infty \tag{10}$$

(10) and the asymptotic formula for eigenvalues $\lambda_n = O(n^{2/m})$ for $n\to\infty$, give

$$\sum_{k=1}^{n}[\varphi_k^2(x_0)-\varphi_k^2(y_0)] = O(n^{1-1/m}) \quad \text{for } n\to\infty. \tag{11}$$

If we denote

$$a_n = \varphi_n^2(x_0)-\varphi_n^2(y_0), \quad b_n = \lambda^{1/4+m/2}, \quad c_n = \lambda_n^{1/2} \tag{12}$$

series (8) fulfils the assumptions of Lemma 1 for $\alpha = \frac{1}{2m}$. Thus (8) converges.

Let $p>\frac{m}{2}$, $p\in N$, $\varrho\in(0,\infty)$ be fixed.

If we consider the series

$$\sum_{n=1}^{\infty}\frac{a_n}{(\lambda_n+\varrho)^{p-1/4}}\varrho^{p-m/2}, \tag{13}$$

we can prove the following

Lemma 2. *A sequence of partial sums of the series* (13) *is uniformly bounded.*

Proof. We can begin with the series

$$\sum_{n=1}^{\infty} \frac{a_n}{(\lambda_n+\varrho)^{m/2-1/4}} \tag{14}$$

and its partial sums $\{C_n\}_{n\in N}$:

$$C_n = \sum_{k=1}^{n} \frac{a_k}{(\lambda_k+\varrho)^{m/2-1/4}} = \sum_{k=1}^{n} \frac{a_k}{\lambda_k^{m/2-1/4}} \cdot \left(\frac{\lambda_k}{\lambda_k+\varrho}\right)^{m/2-1/4}$$

$$= D_n\left(\frac{\lambda_n}{\lambda_n+\varrho}\right)^{m/2-1/4} + \sum_{k=1}^{n-1} D_k\left[\left(\frac{\lambda_k}{\lambda_k+\varrho}\right)^{m/2-1/4} - \left(\frac{\lambda_{k+1}}{\lambda_{k+1}+\varrho}\right)^{m/2-1/4}\right],$$

where $\{D_n\}_{n\in N}$ are partial sums of (8). For each $\varrho\in(0,\infty)$ the sequence $\left\{\left(\frac{\lambda_n}{\lambda_n+\varrho}\right)^{m/2-1/4}\right\}$ increases and $\{D_n\}$ is bounded, so that

$$|C_n| \leqslant K\cdot\left[2\left(\frac{\lambda_n}{\lambda_n+\varrho}\right)^{m/2-1/4} - \left(\frac{\lambda_1}{\lambda_1+\varrho}\right)^{m/2-1/4}\right] \tag{15}$$

and $K = \sup\limits_{n\in N}|D_n|$.

From (15) follows, that

$$\sup_{n\in N}|C_n| \leqslant K\left[2-\left(\frac{\lambda_1}{\lambda_1+\varrho}\right)^{m/2-1/4}\right] \leqslant 2K, \quad \text{for any } \varrho\in(0,+\infty). \tag{16}$$

Partial sums of (13)

$$B_n = \sum_{k=1}^{n} \frac{a_k}{(\lambda_k+\varrho)^{p-1/4}}\varrho^{p-m/2} = \sum_{k=1}^{n} \frac{a_k}{(\lambda_k+\varrho)^{m/2-1/4}}\left(\frac{\varrho}{\lambda_k+\varrho}\right)^{p-m/2}$$

$$= C_n\left(\frac{\varrho}{\lambda_n+\varrho}\right)^{p-m/2} + \sum_{k=1}^{n-1} C_k\left[\left(\frac{\varrho}{\lambda_k+\varrho}\right)^{p-m/2} - \left(\frac{\varrho}{\lambda_{k+1}+\varrho}\right)^{p-m/2}\right]$$

are estimated:

$$|B_n| \leqslant \sup_{n\in N}|C_n|\cdot\left(\frac{\varrho}{\lambda_1+\varrho}\right)^{p-m/2} \leqslant 2K \tag{17}$$

because $\left\{\left(\frac{\varrho}{\lambda_n+\varrho}\right)^{p-m/2}\right\}$ decreases for any $\varrho\in(0,\infty)$. It brings the proof Lemma 2 to the end.

Analogically as above we can prove that partial sums of the series

$$\sum_{n=1}^{\infty} \frac{a_n}{(\lambda_n+\varrho)^{p-1/4}} \varrho^{p-m/2} \left(\frac{2\lambda_n \varrho}{\lambda_n^2+\varrho^2}\right)^k, \tag{18}$$

for a fixed $k \in N$ are uniformly bounded with respect to $\varrho \in (0, \infty)$.

LEMMA 3. *For a_n, λ_n, ϱ, p, k, defined as before, the series*

$$\sum_{n=1}^{\infty} \left(\frac{a_n}{(\lambda_n+\varrho)^p}\right) \varrho^{p-m/2} \left(\frac{2\lambda_n \varrho}{\lambda_n^2+\varrho^2}\right)^k \tag{19}$$

converges for each $\varrho \in (0, \infty)$ and

$$\lim_{\varrho\to\infty} \sum_{n=1}^{\infty} \frac{a_n}{(\lambda_n+\varrho)^p} \varrho^{p-m/2} \left(\frac{2\lambda_n \varrho}{\lambda_n^2+\varrho^2}\right)^k = 0 \tag{20}$$

Proof. Let $\{S_n\}$, $\{\sigma_n\}$ be the partial sums of series (18), (19) respectively. The following formula is true

$$S_n = \sigma_n \left(\frac{1}{\lambda_n+\varrho}\right)^{1/4} + \sum_{k=1}^{n-1} \sigma_k \left[\left(\frac{1}{\lambda_k+\varrho}\right)^{1/4} - \left(\frac{1}{\lambda_{k+1}+\varrho}\right)^{1/4}\right] \tag{21}$$

Because the sequence $\left\{\left(\frac{1}{\lambda_n+\varrho}\right)^{1/4}\right\}$ decreases for all $\varrho \in (0, \infty)$, which together with (21) implies

$$|S_n| \leqslant \sup_{n\in N} |\sigma_n| \left(\frac{1}{\lambda_1+\varrho}\right)^{1/4} \tag{22}$$

for any $n \in N$ and $\varrho \in (0, \infty)$. This equality and the uniform boundedness of $\{\sigma_n\}$ imply Lemma 3.

§ 3. Formula for asymptotic distribution of eigenvalues and eigenfunctions of the operator Δ^{2r}, $r \geqslant 1$.

For clarity we will be assume $|r = 1|$ in this section. The results and proofs can be extended to the case $r > 1$ without difficulty. In the paper [3] the following estimation for eigenvalues of Problem (1)

$$\lambda_n = 4\pi \left[\frac{\Gamma\left(\frac{m}{2}+1\right)}{\mu(D)}\right]^{2/m} n^{2/m} + o(n^{2/m}) \tag{23}$$

and the equality

$$\sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\lambda_n+\varrho)^{k+1}} = \frac{1}{k!}\left\{(2\pi)^{-m}\Gamma\left(k+1-\frac{m}{2}\right)\varrho^{m/2-k-1} - \Phi_k(x,\varrho)\right\} \tag{24}$$

were obtained, where $\{\lambda_n\}$ and $\{\varphi_n\}$ are eigenvalues and eigenfunctions of Problem (1), $\varrho \neq -\lambda_n$ for $n = 1, 2, \ldots$; $k \in N$ and $k \geqslant \left[\frac{m}{2}\right]$, Φ_k is a continuous function of $x \in D$ for each fixed $\varrho > 0$, such that

$$\int_D |\Phi_k(x,\varrho)|\,dx = o(\varrho^{m/2-k-1}) \quad \text{for } \varrho \to \infty . \tag{25}$$

The purpose of this note is to educe the following formula

$$\sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\lambda_n^2+\varrho^2)^{l+1}} = C\cdot\varrho^{m/2-2(l+1)} - \Psi_l(x,\varrho) \tag{26}$$

where $\{\lambda_n^2\}$ and $\{\varphi_n\}$ are eigenvalues and eigenfunctions of Problem (2), $l \geqslant \left[\frac{m}{4}\right]$, $l \in N$, Ψ_l is a continuous function of $x \in D$ for each fixed $\varrho > 0$ such that

$$\int_D |\Psi_l(x,\varrho)|\,dx = o(\varrho^{m/2-2(l+1)}) \quad \text{for } \varrho \to +\infty . \tag{27}$$

The proof of an asymptotic equality for Ψ_l is an essential part of the work and authors further attempts are submitted to it. We begin with

LEMMA 4. *For almost all* $x \in D$

$$\Phi_k(x,\varrho) = o(\varrho^{m/2-k-1}) \tag{28}$$

where Φ_k *is defined by* (24).

Proof. (24) yields

$$\Phi_k(x,\varrho) = k!\left\{(2\sqrt{\pi})^{-m}\Gamma\left(k+1-\frac{m}{2}\right)\varrho^{m/2-k-1} - \sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\lambda_n+\varrho)^{k+1}}\right\}. \tag{29}$$

The estimation (cf.[2], p. 25 (41))

$$\sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\lambda_n+\varrho)^{k+1}} = O(\varrho^{m/2-k-1}) \tag{30}$$

implies that there is an $M < \infty$ such that

$$(31)\qquad \left| \sum_{h=1}^{\infty} \frac{\varphi_n^2(x) \cdot \varrho^{k+1-m/2}}{(\lambda_n + \varrho)^{k+1}} \right| < M$$

for each $\varrho \in (0, \infty)$ and $\varrho \neq -\lambda_n$.

By (29) and (31), for each $\varrho \in (0, \infty)$ there exists $L < \infty$ for which

$$(32)\qquad \varrho_n^{k+1-m/2} |\Phi_k(x, \varrho)| = \left| k! (2\sqrt{\pi})^{-m} \Gamma\left(k+1-\frac{m}{2}\right) - \sum_{n=1}^{\infty} \frac{\varphi_n^2(x) \varrho^{k+1-m/2}}{(\lambda_n + \varrho)^{k+1}} \right| \leqslant L.$$

It follows from the assumptions about Φ_k that for each sequence $\{\varrho_n\}$ $\varrho_n \to \infty$ the sequence $\{\varrho_n^{k+1-m/2} \Phi_k(x, \varrho_n)\}$ fulfils the assumptions of the Lebesgue theorem. This gives, because of (25), the condition:

$$(33)\qquad 0 = \lim_{n|\infty} \varrho_n^{k+1-m/2} \int_D |\Phi_k(x, \varrho_n)| \, dx = \lim_{n|\infty} \int_D \varrho_n^{k+1-m/2} |\Phi_k(x, \varrho_n)| \, dx$$

which implies

$$(34)\qquad \lim_{\varrho|\infty} \varrho^{k+1-m/2} |\Phi_k(x, \varrho)| = 0$$

for almost all $x \in D$, which ends the proof of Lemma 4.

Denote

$$(35)\qquad D_0 := \{x \in D \colon \lim_{\varrho|\infty} \varrho^{k+1-m/2} \Phi_k(x, \varrho) = 0\}.$$

As a consequence of the lemmas presented above, we can formulate:

THEOREM 2. *For sequences of eigenvalues* $\{\lambda_n^2\}$ *and eigenfunctions* $\{\varphi_n\}$ *of the Problem* (2), *for* $l \in N$, $l \geqslant \left[\frac{m}{4}\right]$ *and for each* $x_0, y_0 \in D$ *the following equality*

$$(36)\qquad \sum_{n=1}^{\infty} \frac{\varphi_n^2(x_0) - \varphi_n^2(y_0)}{(\lambda_n^2 + \varrho^2)^{l+1}} = O(\varrho^{m/2-2(l+1)}) \quad \text{for } \varrho \to \infty!$$

Proof. (25) and (35) imply for any $x_0, y_0 \in D_0$

$$(37)\qquad \sum_{n=1}^{\infty} \frac{\varphi_n^2(x_0) - \varphi_n^2(y_0)}{(\lambda_n + \varrho)^{l+1}} = \frac{1}{l!} [\Phi_l(x_0, \varrho) - \Phi_l(y_0, \varrho)]$$

and this, accordingly to (29), gives

$$(38)\qquad \sum_{n=1}^{\infty} \frac{\varphi_n^2(x_0) - \varphi_2^2(y_0)}{(\lambda_n - \varrho)^{l+1}} = O(\varrho^{m/2-l-1}) \quad \text{for } \varrho \to \infty.$$

We can transform the expression in (36):

$$(39)\quad \sum_{n=1}^{\infty} \frac{\varphi_n^2(x_0)-\varphi_n^2(y_0)}{(\lambda_n^2+\varrho^2)^{(l+1)}}\,\varrho^{2(l+1)-m/2} = \sum_{n=1}^{\infty} \frac{\varphi_n^2(x_0)-\varphi_n^2(y_0)}{(\lambda_n+\varrho)^{2(l+1)}}\,\frac{(\lambda_n+\varrho)^{2(l+1)}}{(\lambda_n^2+\varrho^2)^{(l+1)}}\,\varrho^{2(l+1)-m/2}$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{(\lambda_n+\varrho)^{2(l+1)}}\,\varrho^{2(l+1)-m/2}\left[1+\frac{2\lambda_n\varrho}{\lambda_n^2+\varrho^2}\right]^{l+1}$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{(\lambda_n+\varrho)^{2(l+1)}}\,\varrho^{2(l+1)-m/2}\left[1+\binom{l+1}{1}\frac{2\lambda\varrho}{\lambda_n^2+\varrho^2}+\ldots+\left(\frac{2\lambda_n\varrho}{\lambda_n^2+\varrho^2}\right)^{l+1}\right]$$

$$= \sum_{n=1}^{\infty} \frac{a_n\varrho^{2(l+1)-m/2}}{(\lambda_n+\varrho)^{2(l+1)}} + \binom{\lambda+1}{1}\sum_{n=1}^{\infty} \frac{a_n\varrho^{2(l+1)-m/2}}{(\lambda_n+\varrho)^{2(l+1)}}\cdot\frac{2\lambda_n\varrho}{\lambda_n^2+\varrho^2}$$

$$\ldots + \sum_{n=1}^{\infty} \frac{a_n\varrho^{2(l+1)-m/2}}{(\lambda_n+\varrho)^{2(l+1)}}\left(\frac{2\lambda_n\varrho}{\lambda_n^2+\varrho^2}\right)^{l+1},$$

where $a_n := \varphi_n^2(x_0)-\varphi_n^2(y_0)$.

Lemma 4 and Lemma 3 with substitutions $p = 2(l+1)$, $k = 1, \ldots, l+1$ give the proposition of the Theorem 2.

Note, that $\varrho > 0$ the series

$$(40)\qquad \sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\lambda_n^2+\varrho^2)^{l+1}}$$

is uniformly convergent with respect to $x \in D$. We can get, after a transformation similar to (39),

$$\sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\lambda_n^2+\varrho^2)^{l+1}} = \sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\lambda_n+\varrho)^{2(l+1)}}\,\frac{(\lambda_n+\varrho)^{2(l+1)}}{(\lambda_n^2+\varrho^2)^{l+1}}$$

$$= \sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\lambda_n+\varrho)^{2(l+1)}}\left(1+\frac{\lambda_n\varrho}{\lambda_n^2+\varrho^2}\right)^{l+1}.$$

The series

$$\sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\lambda_n+\varrho)^{2(l+1)}}$$

for any $\varrho > 0$ is uniformly convergent for $x \in D$ and

$$\left| \binom{l+1}{1} \frac{2\lambda_n \varrho}{\lambda_n^2+\varrho^2} + \ldots + \left(\frac{2\lambda_n \varrho}{\lambda_n^2+\varrho^2} \right)^{l+1} \right| < 2^{l+1}$$

These two facts yield the uniform convergence of the series (40) in D (cf. [5] p. 370).

Let us define

$$H_l(x,\varrho) := \sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\lambda_n^2+\varrho^2)^{l+1}} \tag{41}$$

and integrate this equality in D. The result is

$$\int_D H_l(x,\varrho)\,dx = \sum_{n=1}^{\infty} \int_D \frac{\varphi_n^2(x)}{(\lambda_n^2+\varrho^2)^{l+1}}\,dx = \sum_{n=1}^{\infty} \frac{1}{(\lambda_n^2+\varrho^2)^{l+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(\lambda_n+\varrho)^{2(l+1)}} \left[\binom{l+1}{1} \frac{2\lambda_n \varrho}{\lambda_n^2+\varrho^2} + \ldots + \left(\frac{2\lambda_n \varrho}{\lambda_n^2+\varrho^2} \right)^{l+1} \right]$$

$$\leqslant 2^{l+1} \sum_{n=1}^{\infty} \frac{1}{(\lambda_n+\varrho)^{2(l+1)}}.$$

[2], formula (33) gives

$$\int_D H_l(x,\varrho)\,dx \leqslant 2^{l+1} \frac{1}{[2(l+1)]!} \left\{ \frac{\mu(D)\Gamma\left(2l+2-\frac{m}{2}\right)}{(2\sqrt{\pi})^m \varrho^{2l+2-m/2}} - o(\varrho^{m/2-2(l+1)}) \right\}$$

and this means

$$\int_D H_l(x,\varrho)\,dx = \mu(D) A \varrho^{\alpha} - o(\varrho^{\alpha}) \tag{42}$$

where $\alpha \leqslant \frac{m}{2} - 2(l+1)$.

Our next purpose is to determine the value of α. The Tauber's theorem will be used (cf. [8], p. 433). Let the number of eigenvalues of Problem (2), not exceeding ϱ^2 be denoted as $N(\varrho^2)$. Note, that $N(\varrho^2) = n$ for $\lambda_n^2 \leqslant \varrho^2 < \lambda_{n+1}^2$. Thus

$$\int_{\lambda_n^2}^{\lambda_{n+1}^2} \frac{N(t)}{(t+\varrho^2)^{l+2}}\,dt = \frac{n}{l+1} \left\{ \frac{1}{(\lambda_n^2+\varrho^2)^{l+1}} - \frac{1}{(\lambda_{n+1}^2+\varrho^2)^{l+1}} \right\}$$

and therefore

$$(43)\qquad \int_0^{+\infty} \frac{N(t)}{(t+\varrho^2)^{l+2}}\,dt = \frac{1}{l+1}\sum_{n=1}^{\infty}\left\{\frac{n}{(\lambda_n^2+\varrho^2)^{l+1}} - \frac{n}{(\lambda_{n+1}^2+\varrho^2)^{l+1}}\right\} = \frac{1}{l+1}\sum_{n=1}^{\infty}\frac{1}{(\lambda_n^2+\varrho^2)^{l+1}}.$$

(42) yields

$$(44)\qquad \sum_{n=1}^{\infty}\frac{1}{(\lambda_n^2+\varrho^2)^{l+1}} \sim \frac{A\mu(D)}{\varrho^{-\alpha}\alpha} \quad \text{for } \varrho\to\infty$$

In virtue of Tauber's theorem, (44) gives

$$N(\varrho^2) = \frac{A\mu(D)\Gamma(l+2)}{\Gamma\left(l+2+\frac{\alpha}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right)}\,\varrho^{2(l+1+\alpha/2)} + o(\lambda^{2(l+1+\alpha/2)})$$

or, if we suppose $\varrho^2 = \lambda_n^2$, $N(\lambda_n^2) = n$ and we get

$$n = \left[\frac{\Gamma\left(l+1+\frac{\alpha}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right)}{A\mu(D)\Gamma(l+1)}\right]^{-1} \lambda_n^{2(l+1+\alpha/2)} + o(\lambda_n^{2(l+1+\alpha/2)})$$

Therefore

$$(45)\qquad \lambda_n = \left[\frac{\Gamma\left(l+2+\frac{\alpha}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right)}{A\mu(D)\Gamma(l+2)}\right]^{\frac{1}{2(l+1+\alpha/2)}} n^{\frac{1}{2(l+1+\alpha/2)}} + o(n^{\frac{1}{2(l+1+\alpha/2)}}).$$

Comparison of (23) and (45) allows us to write

$$\frac{1}{2\left(l+1+\frac{\alpha}{2}\right)} = \frac{2}{m}$$

which means

$$(46)\qquad \alpha = \frac{m}{2} - 2(l+1).$$

Summing up, the following equality

$$(47)\qquad \int_D H_l(x,\varrho)\,dx = \mu(D)A\varrho^{2(l+1)-m/2} + o(\varrho^{2(l+1)-m/2})$$

was proved, and Theorem 2 for $x_0, y_0 \in D$ gives

$$(48)\qquad \lim_{\varrho|\infty} \varrho^{m/2-2(l+1)}(H_l(x_0,\varrho) - H_l(y_0,\varrho)) = 0.$$

These two results imply, for $x \in D$,

$$H_l(x, \varrho) = C\varrho^{2(l+1)-m/2} - \psi_l(x, \varrho), \tag{49}$$

where the constant C does not depend on ϱ. To prove (27), we need:

LEMMA 5. *If:*

1° Ψ_l *is a continuous function for* $x \in D$

2° $\int\limits_D \Psi_l(x, \varrho)dx = o(\varrho^\alpha)$ *for* $\varrho \to \infty$

3° $\lim\limits_{\varrho|\infty} \varrho^{-\alpha}((\Psi_l(x_0, \varrho) - \Psi_l(y_0, \varrho)) = 0$ *for* $x_0, y_0 \in D_0 \subset D$

then for each $x \in D$

(5°) $\Psi_l(x, \varrho) = o(\varrho^\alpha)$ for $\varrho \to \infty$.

Proof. We fix $\bar{x}_0 \in D_0$. Let us assume the contrary i.e. that $\varrho^{-\alpha}\Psi_l(\bar{x}_0, \varrho)$ does not tend to zero when $\varrho \to \infty$. Thus a sequence $\{\varrho_n\}$ exists, for which $\varrho_n \to \infty$ when $n \to \infty$ and

$$\lim_{n|\infty} \varrho_n^{-\alpha}\Psi_l(\bar{x}_0, \varrho_n) = g_n,$$

where $g \neq 0$.

It follows from 1° that for each $x_0 \in D_0$, $\lim\limits_{n|\infty} \varrho^{-\alpha}\Psi_l(x_0, \varrho_n) = g$. Formulas (49) and (41) imply

$$\Psi_l(x, \varrho) = \sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\lambda_n^2 + \varrho^2)^{l+1}} - C\varrho^\alpha$$

If we note that

$$\sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\lambda_n^2 + \varrho^2)^{l+1}} = o(\varrho^\alpha)$$

there is $L < \infty$ such that

$$|\Psi_l(x, \varrho)| < L\varrho^\alpha .$$

Then

$$|\varrho_n^{-\alpha}\Psi_l(x, \varrho_n)| \leqslant L \quad \text{for } n \in N .$$

It means that the sequence $\{\varrho_n^{-\alpha}\Psi_l\}$ fulfils the assumptions of the Lebesgue theorem. Thus 2° yields

$$0 = \lim_{n|\infty} \int\limits_D \varrho_n^{-\alpha}\Psi_l(x, \varrho_n)dx = \int\limits_D \lim_{n|\infty} \varrho_n^{-\alpha}\Psi_l(x, \varrho_n)dx = \int\limits_D g\,dx = g\mu(D)$$

D has a non-zero measure, then $g = 0$, which contradicts the assumption $g \neq 0$. That contradiction proves

$$\Psi_l(x, \varrho) = o(\varrho^{2(l+1)-m/2}) \quad \text{when } \varrho \to \infty$$

and, as a simple consequence, we obtain (27).

The proof of the equality (27) completes the proof of (26), which is the main purpose of this note.

Remark. An application of (26) to the inverse Sturm-Liouville problem for Problem (2) will be given in the next paper.

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