UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXVI

1987

On the Inverse Problem of the Sturm-Liouville Type for some Class of the Partial Differential Equations of the Fourth Order

by Stanisława POSTAWA

Introduction. In the paper [7] a formula was introduced the asymptotic distribution of eigenvalues and eigenfunctions for the problem

$$\Delta^2 u-\lambda^2 u=0 \quad \text{and} \quad u\in W_2^4(bc) \tag{1}$$

The purpose of the present paper is to give an answer to the question: what is a sufficient condition which should be imposed on the sequence of eigenvalues of the problem

$$\Delta^2 u-\lambda^2 u+qu=0 \quad \text{and} \quad u\in W_2^4(bc) \tag{2}$$

so that the function $q(x)\equiv 0$ in D.

§ 1. Some properties of the trace of a linear operator.

In this section we shall give a definition of the trace of a linear operator and we shall cite some lemmas and theorems, which will be used in the sequel.

The trace of a linear operator A on the Hilbert space H is the series

$$\sum_{n=1}^{\infty}(Ax_n, x_n) \tag{3}$$

where $\{x_n\}$ denotes any complete orthonormal system of vectors in H. (cf. [6], p. 125).

If the series (3) is convergent for any $\{x_n\}$, we say that A has a finite trace. In the sequel the trace of the operator A will be denoted by $S(A)$.

LEMMA 1. *If the operator A has a finite trace, then the sum* (3) *is independent of* $\{x_n\}$ *and the series* (3) *converges absolutely for each* $\{x_n\}$.

LEMMA 2. *If the operator A is positive, then the sum* (3) *has the same value* (*finite or not*) *for any* $\{x_n\}$ (cf. [6], p. 126—127).

We denote by

$$T:=\Delta^2 \text{ and by } D(T):=W_2^2(bc)\cap W_2^4(bc).$$

$\overline{D(T)}=L_2(D)$ and T is a self-adjoint and positive operator.

Let us denote by $V\colon D(T)\ni u\to V(u)=q\cdot u\in L_2(D)$, where q is a continuous function in D. V is a self-disjoint and bounded operator on $L_2(D)$.

By $T_\varrho := T+\varrho^2E$, where E is the identity operator on $L_2(D)$, and ϱ is a real positive number such that the operators T_ϱ and $T_\varrho+V$ are positively defined operators on $L_2(D)$. T_ϱ^{-1} and $(T_\varrho+V)^{-1}$ are completely continuous operators (cf. [1]).

We shall denote by $\{\lambda_n^2\}_{n\in N}$ and $\{\mu_n^2\}_{n\in N}$ the increasing sequences of all eigenvalues of T and $T+V$, respectively, and by $\{\varphi_n\}$ and $\{\psi_n\}$ the corresponding orthonormal sequences of eigenfunctions.

It follows from the assumptions of T and $T+V$ that $\lim\mu_n=\lim\lambda_n=\infty$, and $\{\varphi_n\}$ and $\{\psi_n\}$ form the complete orthornormal systems in $L_2(D)$ (cf. [8], p. 579).

The sequence of eigenvalues $\{\lambda_n\}$ of the operator T satisfies the following condition

$$\lambda_n^2 = Cn^{4/m}+o(n^{4/m}) \quad \text{when } n\to+\infty \tag{4}$$

where C is a positive constant independent of n. (cf. [7], p. 9). Therefore series

$$\sum_{n=1}^{\infty}\frac{1}{(\lambda_n^2+\varrho^2)^{l+1}} \tag{5}$$

is convergent for $\varrho>0$ and $l\geqslant\left[\frac{m}{4}\right]$.

We shall prove the following

THEOREM 1. *If* 1° $\lambda_n^2 = Cn^{4/m}+o(n^{4/m})$

2° $\displaystyle\sum_{n=1}^{\infty}\frac{|\lambda_n^2-\mu_n^2|}{n}$ *is convergent,*

then

$$\lim_{\varrho|\infty}\varrho^{2(l+1)+m/2}S(T_\varrho^{-l}-(T_\varrho+V)^{-l})=0 \tag{6}$$

where l denote the natural number such that $l\geqslant\left[\frac{m}{4}\right]$.

Proof. From the assumption 1° we get (5). Hence by Lemma 4 of paper [3] follows that $T_\varrho^{-l}-(T_\varrho+V)^{-l}$ has a finite trace for $\varrho>0$ and $l\geqslant\left[\frac{m}{4}\right]$ and we get

$$S\{T_\varrho^{-l}-(T_\varrho+V)^{-l}\} = \sum_{n=1}^{\infty}\left\{\frac{1}{(\lambda_n^2+\varrho^2)^l}-\frac{1}{(\mu_n^2+\varrho^2)^l}\right\} \tag{7}$$

Using (7) we need to prove that

$$\lim_{\varrho\to\infty}\varrho^{2(l+1)-m/2}\sum_{n=1}^{\infty}\left\{\frac{1}{(\lambda_n^2+\varrho^2)^l}-\frac{1}{(\mu_n^2+\varrho^2)^l}\right\}=0. \tag{8}$$

Let us observe that

$$(9)\qquad \sum_{n=1}^{\infty} \varrho^{2(l+1)-m/2}\left\{\frac{1}{(\lambda_n^2+\varrho^2)^l}-\frac{1}{(\mu_n^2+\varrho^2)^l}\right\} = \sum_{n=1}^{\infty} \frac{\varrho^{2(l+1)-m/2}}{(\lambda_n^2+\varrho^2)^l}\cdot\frac{(\mu_n^2+\varrho^2)^l-(\lambda_n^2+\varrho^2)^l}{(\mu_n^2+\varrho^2)^l}$$

$$= \sum_{n=1}^{\infty} \frac{\mu_n^2-\lambda_n^2}{n}\cdot\frac{n}{(\lambda_n^2+\varrho^2)^{m/4}}\cdot\left(\frac{\varrho^2}{\lambda_n^2+\varrho^2}\right)^{l-m/4}\frac{[\varrho^2(\mu_n^2+\varrho^2)^{l-1}+ .. +\varrho^2(\lambda_n^2+\varrho^2)^{l-1}]}{(\mu_n^2+\varrho^2)^l}.$$

Since the sequences

$$\left\{\frac{n}{(\lambda_n^2+\varrho^2)^{m/4}}\right\},\ \left\{\frac{\varrho^2}{\lambda_n^2+\varrho^2}\right\}^{l-m/4}\ \left\{\frac{\varrho^2(\mu_n^2+\varrho^2)^{l-1}+\ldots+\varrho^2(\lambda_n^2+\varrho^2)^{l-1}}{(\mu_n^2+\varrho^2)^l}\right\}$$

are uniformly bounded in ϱ for $\varrho > 0$ and $n \in N$, from the last equality follows that the series

$$M\sum_{n=1}^{\infty}\frac{|\lambda_n^2-\mu_n^2|}{n}$$

is an absolute majorant of the series

$$\sum_{n=1}^{\infty} \varrho^{2(l+1)-m/2}\left\{\frac{1}{(\lambda_n^2+\varrho^2)^l}-\frac{1}{(\mu_n^2+\varrho^2)^l}\right\},$$

where M is a positive constant independent of $\varrho > 0$ and $n \in N$. Consequently this series is absolutely and uniformly convergent in ϱ for $\varrho > 0$. Therefore from this we get (8). From (8) by (7) we obtain (6).

§ 2. Dependence of the function q on the eigenvalues of problem (2)

THEOREM 2. *If:*

1° *q is a continuous and bounded function in D*

2° *the series* $\sum_{n=1}^{\infty}\frac{|\lambda_n^2-\mu_n^2|}{n}$ *is convergent,*

then

$$\int_D q(x)\,dx = 0.$$

Proof. Let ϱ be a real positive number such that $\lambda_1^2+\varrho^2 > \|V\|$. Then

$$(10)\qquad T_\varrho^{-l}-(T_\varrho+V)^{-l} = B_\varrho+F_\varrho,$$

where

$$B_\varrho = T_\varrho^{-l}VT_\varrho^{-1}+\ldots+T_\varrho^{-l}VT_\varrho^{-1}$$

and

$$|S(F_\varrho)| \leqslant \frac{M}{\lambda_1^2+\varrho^2} \sum_{n=1}^{\infty} \frac{1}{(\lambda_n^2+\varrho^2)^{l+1}}$$

(cf. [3], lemma 4)
By (44) and (46) from [7] we get,

(11) $$\lim_{\varrho\to\infty} \varrho^{2(l+1)-m/2} S(F_\varrho) = 0 .$$

We express the trace of the operator B_ϱ by the sequence $\{\varphi_n\}$. We obtain

(12) $$S(B_\varrho) = l \sum_{n=1}^{\infty} \frac{(V\varphi_n, \varphi_n)}{(\lambda_n^2+\varrho^2)^{l+1}} .$$

By the definition of the operator V and by the uniform convergence of the series in the formula (26) of paper [7] with respect to $x \in D$, the equality (12) can be written in the form

(13) $$S(B_\varrho) = l\left(V, \sum_{n=1}^{\infty} \frac{\varphi_n^2}{(\lambda_n^2+\varrho^2)^{l+1}}\right).$$

By (26) of paper [7] the equality (13) take the form

(14) $$S(B_\varrho) = lC\varrho^{m/2-(l+1)}(V, 1) - \int_D \Psi_l(x, \varrho) q(x)\, dx .$$

Since

$$\int_D |\Psi_l(x, \varrho) q(x) dx| \leqslant \max_{x\in D} |q(x)| \int_D \Psi_l(x, \varrho)|\, dx ,$$

by (27) of paper [7] we get

(15) $$\int_D |\Psi_l(x, \varrho)|\, dx = o(\varrho^{m/2-2(l+1)}), \quad \text{when } \varrho \to \infty .$$

By (6), (10), (11), (14) and (15) we obtain

(16) $$0 = \lim_{\varrho\to\infty} \varrho^{2(l+1)-m/2} [S(B_\varrho)+S(F_\varrho)] = lC(V, 1) .$$

The equality (16) is possible only when

(17) $$(V, 1) = 0 .$$

It is obvious that (17) is equivalent to the equality

(18) $$\int_D q(x)\, dx = 0 .$$

The proof of Theorem 2 is completed.

THEOREM 3. *Under the assumptions of Theorem 2, if* $\sigma(x) = 0$, *then* $q(x) \equiv 0$ *in* D.

Proof. It is known (see [5] or [8]) that

$$\mu_1^2 = \min J[\varphi] = \min_{\varphi \in K} \int_D [(\Delta\varphi(x))^2 + q(x)\varphi^2(x)]\,dx \tag{19}$$

where K is the class of functions φ which are of class C_σ^2 in D, satisfying the condition $\int_D \varphi^2(x)\,dx = 1$.

Putting $\varphi_1 = [\mu(D)]^{-1/2}$ in (19), by (18) we see that

$$J[\varphi_1] = 0 \tag{20}$$

It follows that $\mu_1^2 = 0$ and $\varphi_1 = [\mu(D)]^{-1/2}$.

Therefore the function $\varphi_1(x) = [\mu(D)]^{-1/2}$ satisfies (2) for $\lambda = 0$ and $\sigma(x) = 0$ (cf. [5], p. 500).

It follows that $q(x) \equiv 0$ in D.

Remark. The results of the Theorem 3 may be transferred to the following case:

$$\Delta^{2r}u - \lambda^{2r}u + qu = 0 \quad \text{and } u \in W_2^{2r}(bc) \tag{21}$$

but the assumption 2° of the Theorem 2 now takes the form:

the series $\sum_{n=1}^{\infty} \frac{|\lambda_n^{2r} - \mu_n^{2r}|}{n}$ is convergent.

References

[1] Ю. М. Верезанский, *Разложение по собственным функциям самосопряженных операторов*, Киев (1965).

[2] J. Bochenek, *Zależność równania różniczkowego cząstkowego od wartości własnych odpowiedniego zagadnienia*, Zeszyty naukowe Politechniki Krakowskiej z. 15 (1968), 3—54.

[3] —, *On the inverse problem of the Sturm-Liouville type for a linear elliptic partial differential equation with constant coefficients of the second order*, Ann. Polon. Math., 24 (1971), 331—341.

[4] —, *Global inverse problem of the Sturm-Liouville type for a linear elliptic partial differential equation of the second order*, Ann. Polon. Math. *40 (1983), 271—281.*

[5] R. Courant and D. Hilbert, *Methods of mathematical physics* II, New York 1953.

[6] И. Ц. Гохберг и М. Г. Крейн, *Введение в теорию линейных несамосопряженных операторов в гильбертовом пространстве*, Москва, 1965.

[7] St. Postawa, *An asymptotic distribution of eigenvalues and eigenfunctions of the operator* Δ^{2r}, $r \geqslant 1$, *this fasc., 137—148.*

[8] В. И. Смирнов, *Курс высшей математики* **X**, Москва, 1959.

Received November 9, 1981.