

One Dimensional Maps without Asymptotic Periodicity Property

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1. In biological applications the following functions are frequently used to describe the behaviour of the biological systems.

$$(1) \quad f_t(x) = tx(1-x), \quad x \in [0, 1], \quad t \in [0, 4].$$

These functions generate a one parameter family $F_t: N \times [0, 1] \rightarrow [0, 1]$ of discrete semi-dynamical systems on $I = [0, 1]$ defined by

$$F_t(n, x) = f_t^n(x),$$

where f^n denotes the n -th iterate of f : $f^0 =$ the identity map, $f^{n+1} = f^n \circ f$ (see [1], [7] for definitions). It is well known ([4], [6], [10]) that for the values of t from the interval $[0, t_c]$, approx. $t_c = 3.570$, f_t has the AP-property i.e. for every $x \in I$ there is a periodic point $p \in I$ such that $\lim_{n \rightarrow \infty} (f^n(x) - f^n(p)) = 0$ ([4], [6]). For $t > t_c$ situation complicates and still it is not fully known. For $t = 3.627$ f_t has 6-periodic point and there is chaos, for $t = 3.83$ f_t has 3-periodic point ([4]).

In this note we present more detailed discussion of the situation described above. In particular we will show that there exist numbers s_1 and u , $s_1 < 3.6786 < u < 3.83$, such that for $t \in (s_1, 3.6786]$ f_t has 6-periodic points, and for $t < u$ f_t has no 3-periodic points. Generally speaking, periodic points with high periods i.e. with periods 3, 6, 12, 24, ... disappear as $t \searrow t_c$.

We also give numerical conditions for pseudo-dyadicity of the function f^2 ([3]) whether f is like to the function (1), and for the existence of 3-periodic points ([2], [3], [4], [6]) for piecewise linear functions. Clearly results obtained here for the interval $[0, 1]$ are also valid for an interval $[0, A]$, $A \leq \infty$.

2. We will assume in the sequel without any further mention that $f: I \rightarrow I$ is a continuous function satisfying

$$(2) \quad \begin{aligned} f(0) &= f(1) = 0, \\ f &\text{ increases in } [0, c] \text{ and decreases in } [c, 1], \quad 0 < c < 1, \\ f &\text{ is of the class } C^3 \text{ in } I \setminus \{c\}, \\ f'(0) &> 1, \end{aligned}$$

and either

$$(3) \quad f \text{ is linear (affine) in } [0, c] \text{ and } [c, 1],$$

or

$$(4) \quad S(f)(x) < 0, \quad \text{for } x \neq c,$$

where

$$S(f)(x) = \frac{f'''(x)}{f''(x)} - \frac{3}{2} \left(\frac{f'''(x)}{f''(x)} \right)^2$$

is the Schwarzian derivative of f .

Conditions (2) and (4) are satisfied by functions appearing in majority mathematical models describing behaviour of biological populations ([9]). In [9] and [7] some elementary properties of $S(f)$ are given. Conditions (2) and (3) are also satisfied in the model considered in [5].

We will use some results from [7] and [8] summarized in the following lemma.

LEMMA 1. f has exactly one non-zero fixed point $p \in I$. The point p is asymptotically stable, iff, p attracts all points from $(0, 1)$ ($f^n(x) \rightarrow p$, as $n \rightarrow \infty$, for every $x \in (0, 1)$), iff, either $f'(p) > -1$ provided (3) holds, or $f'(p) \geq -1$ provided (4) holds. If p is unstable, then there are exactly two points p_1, p_2 satisfying $0 < p_1 < p < p_2$ and $f(p_1) = p_2$, $f(p_2) = p_1$. The orbit $o(p_1) = o(p_2) = \{p_1, p_2\}$ is asymptotically stable, iff, $o(p_1)$ attracts all points from $(0, 1) \setminus \{x: f^k(x) = p, \text{ for some } k\}$.

For $x \in [0, f(c)]$ we will denote by x_l, x_r , the uniquely defined points of I such that $f(x_l) = f(x_r) = x$, $x_l \leq c \leq x_r$. Similarly one can define x_{lr} as $(x_l)_r$, etc.

Put $g = f^2$.

LEMMA 2. Let $c < p$ and $g(c) < c$. Then:

p is unstable,

$$c_l < p_l < p_1 < c < p < c_r < p_2 < p_{lr},$$

$$g'(p) > 1$$

g has a local minimum at c and maxima at c_l and c_r , which are equal to $f^2(c)$ and $f(c)$ respectively (see Fig.).

Proof. As $g'(0) = f'(0)^2 > 1$ and $g(c) < c$, then there is a point p_1 such that $0 < p_1 < c$ and $g(p_1) = p_1$. Put $p_2 = f(p_1)$. It is obvious that $c_r < p_2$ and $g(p_2) = p_2$. Hence p does not attract p_2 , which by Lemma 1 implies that p is unstable. Again by Lemma 1 $g'(p) = f'(p)^2 > 1$. The remaining statements are easy to see.

We will consider functions f which have not the AP-property. From [4] it follows that if for some n f^n has 3-periodic point, then f has not the AP-property. We say the orbit $o(x)$ is strictly turbulent if the limit set $\omega(x)$ is infinite and does not contain periodic points [3]). If there is a strictly turbulent trajectory, then f has not AP-property. The consequence of Lemma 1 is the following:

LEMMA 3. If f has not the AP-property, then $g(c) < c$.

Proof. If the inequality $g(c) \geq c$ held, then it would be $g(c_r) \geq c_r$. The points p_1, p_2 satisfy the condition $c < p_1 < p < p_2 < c_r$ and hence $0 < g'(p_1) = g'(p_2) < 1$, that means the asymptotic stability of the orbit $\{p_1, p_2\}$, and by Lemma 1 f would have the AP-property.

In the next section we will use following conditions:

- (5) g is pseudo-dyadic, i.e. there are two open disjoint intervals $K, L \subset I$, $K \cup L \subset g(K) \cap g(L)$ ([2]).
- (6) $f^3(c) \leq p$, $c < p$,
- (7) f has 3-periodic point (by [4] this is equivalent to the property that f has n -periodic point for every positive integer n).
- (8) $f^3(a) \leq a$ for some $a \in [c, p)$.

Remark. In [2] it has been proved (without assumptions (2), (3), (4)), that (7) implies (5) but we do not use it here. Obviously (8) implies (7).

LEMMA 4. If f satisfies one of the conditions (5)—(8), then f has not the AP-property.

Proof. Assume (5) holds. Then, as shown in [3], there is a strictly turbulent trajectory.

Assume (6) holds. Then $g(c) < c$ and by Lemma 2, $g(c) \leq p_1$. On the other hand, $g(p_1) = p_1 < c$, and by continuity of g there is a point $a \in (p_1, c]$ such that $g(a) = p_1 < a$. Hence $g^3(a) = p$, but since $g'(p) > 1$, there is a point $b \in (a, p)$ such that $g^3(b) < b$, which implies that there is a 3-periodic point of g in (a, b) . Hence g and also f has not the AP-property. If (7) or (8) holds the proof is obvious.

LEMMA 5. Assume that f has not the AP-property and

$$(9) \quad f^3(c) > p.$$

Then:

- (a) f has not $(2n+1)$ -periodic points, $n = 1, 2, 3, \dots$,
- (b) there is a point $c_1 \in (0, 1)$ and a function $f_1: I \rightarrow I$ satisfying conditions (2), and either (3) or (4), such that, f has $2n$ -periodic point, if and only if, f_1 has n -periodic point, $n = 1, 2, 3, \dots$,
- (c) $f_1(c_1) = 1$, if and only if, $f^3(c) = p$.

Proof. Lemma 3 implies that $g(c) < c$, hence p is unstable, $c < p$, so inequalities from Lemma 2 hold. (9) implies $g(c) \geq p_l$, $g(c_r) \leq p_r$, which means that

$$f([p_l, p]) \subset [p, p_r]$$

and

$$f([p, p_r]) = [p_l, p].$$

Hence $[p_l, p_r]$ is the invariant set for f and $[p_l, p]$, $[p, p_r]$ are invariant for g . Obviously for $x \in I$, $f(x) \leq f(c)$. If $x < c$ then $x < f(x)$, there is a positive integer $k(x)$ such that $f^{k(x)}(x) \geq c$, otherwise the limit of the sequence $x < f(x) < f^2(x) < \dots < c$ would be

a non-zero fixed point of f different from p . It follows that $[p_l, p_{lr}]$ attracts all points from $(0,1)$, hence it contains all periodic points of f and of g . Now (a) is clear. To prove (b) and (c) it suffices to put for $x \in I$

$$f_1(x) = \frac{1}{p_{lr} - p} (g(p_{lr} - p)x + p) - p,$$

$$c_1 = \frac{c_r - p}{p_{lr} - p}.$$

3. PROPOSITION 1. *The conditions (5) and (6) are equivalent.*

Proof. Observe that if (5) or (6) are satisfied, then by Lemmas 4 and 3 the statement of Lemma 2 holds.

First assume the condition (6). Put $K = (p, c_r)$, $L = (c_r, p_{lr})$. We see that $g(K) = g(L) = (p, f(c))$. (6) implies $f(c) \geq p_{lr}$, otherwise $f(c) < p_{lr}$, $f^2(c) > f(p_{lr}) = p_l$, and since $f^2(c) < c$ then $f^3(c) > f(p_l) = p$. Hence $K \cup L \subset (p, f(c)) = g(K) \cap g(L)$.

Now assume condition (6) is false. We are going to prove that g is not pseudo-dyadic. Otherwise, there are open disjoint intervals K and L satisfying $K \subset g(K)$, $K \subset g(L)$, $\bar{L} \subset g(K)$, $L \subset g(L)$, and moreover $K \cup L \subset [0, f(c)]$. By continuity g has fixed points $a \in K$, $b \in L$. Obviously a and b belongs to the set $\{0, p, p_1, p_2\}$ of all fixed points of g , therefore we will discuss all possible cases.

Case $a = 0$, $b \in \{p, p_1, p_2\}$. We have $0 \in \bar{K} \subset g(\bar{L})$. On the other hand $L \subset (0, f(c)) \setminus K$, and $\inf g(L) \geq \min\{k, g(c)\}$, where k is the right end of K . Since (6) is false, then $g(c) > 0$ and 0 cannot belong to $g(L)$.

Case $a = p_1$, $b = p$. As $K \cap L = \emptyset$ and $g(c) > p_l$, we obtain $K \subset (p_l, p)$ and $g(K) \subset (p_l, p)$. So $g(\bar{K}) \supset \bar{L} \ni p$. Since $p_1 \in \bar{K} \subset g(\bar{K})$ we have obtained that $g(K) \supset (p_1, p)$ contrary to $g(c) > p_l$.

In the case $a = p$, $b = p_2$ the reasoning is similar to the last one.

Case $a = p_1$, $b = p_2$. As shown in the proof of Lemma 5 intervals $[p_1, p]$, $[p, p_l]$ are invariant for g , so for instance cannot be $p_1 \in g(K)g(L)$.

The all cases in which $a = b$ are trivial and so the proof is complete.

PROPOSITION 2. *The conditions (7) and (8) are equivalent.*

Proof As in the proof of Proposition 1 the statement of Lemma 2 holds. First we claim that f^3 increases in $[c, p_{2r}]$. It follows from observations that: $(f^3)'(x) = f'(f^2(x)) \cdot f'(f(x)) \cdot f'(x)$, f decreases in (c, p_{2r}) and $(p_2, f(c))$, increases in $(f^2(c), p_1)$ and moreover for $x \in (c, p_{2r})$, $f(x) \in (p_2, f(c))$ and $f^2(x) \in (f^2(c), p_2)$.

Since $f'(0) > 1$ and f is continuous, (8) implies (7). Assume (7) holds. Let the 3-periodic orbit contains points a_1, a_2, a_3 such that $a_2 = f(a_1)$, $a_3 = f(a_2)$, $a_1 = f(a_3)$. We may also assume $a_1 < a_2$, $a_1 < a_3$. Hence we have $p < a_3$ and $p_l < a_2 < p$. Define $a \in [c, p)$, either $a = a_2$ if $c \leq a_2$, or $a = (a_3)_r$ if $a_2 < c$. So $f^3(a) \leq a$. Moreover for $x \in [p_{2r}, p)$ we have $p < f(x) \leq p_2$ and hence $f^3(x) = g(f(x)) \geq f(x) > x$. So a belongs to the interval $[c, p_2)$ what completes the proof.

COROLLARY. *If the condition (3) holds, then f has 3-periodic point, if and only if, $f^3(c) \leq c$.*

Proof. f^3 restricted to $[c, p_{2r})$ is affine linear. Since $f'(0) > 1$ and p is unstable, then for $x \neq c$ we have $|f'(x)| > 1$. In the proof of Proposition 2 it has been shown that $(f^3)'(x) > 0$ and the linear function $f^3(x) - x$ increases in $[c, p_{2r})$. If $f^3(a) \leq a$ for $a \in [c, p_{2r})$, then $f^3(c) \leq c$.

4. Let $\Delta \subset \mathbb{R}$ be an interval and let for $t \in \Delta$ $f_t: I \rightarrow I$ be the family of functions such that the mapping $F: \Delta \times I \ni (t, x) \rightarrow f_t(x) \in I$ is continuous. Suppose that for every $t \in \Delta$ the function f_t satisfies the conditions (2), and either (3) or (4).

THEOREM. *Let $AP = \{t \in \Delta: f_t \text{ has AP-property}\}$ be non empty set and denote $t_c = \sup AP$. Let t_0 satisfies $f_{t_0}(c_{t_0}) = 1$. Then there are two sequences $\{t_k\}, \{s_k\} \subset \Delta$ such that*

$$t_0 > s_0 > t_1 > s_1 > \dots > t_c,$$

and

$$(10) \quad f_t \text{ has not } (2n+1)2^l\text{-periodic points, for } t \in (t_c, t_k),$$

$$(11) \quad n = 1, 2, 3, \dots, i = 0, 1, 2, \dots, k-1,$$

f_t has $n2^k$ -periodic points for $t \in [s_k, t_k], n = 1, 2, 3, \dots$

First we prove

LEMMA 6. *The function $\Delta \ni t \rightarrow c_t \in I$ is continuous.*

Proof. Let $t_n \rightarrow \hat{t}$, as $n \rightarrow \infty$. It is sufficient to show that every convergent subsequence of $\{c_{t_n}\}$ (denote it also $\{c_{t_n}\}$) tends to $c_{\hat{t}}$. Let $c_{t_n} \rightarrow d$ and fix $x \in I$. By the assumption (2), $f_{t_n}(x) \leq f_{t_n}(c_{t_n})$, so by the continuity of F , $f_{\hat{t}}(x) \leq f_{\hat{t}}(d)$. By uniqueness of $c_{\hat{t}}$, $d = c_{\hat{t}}$.

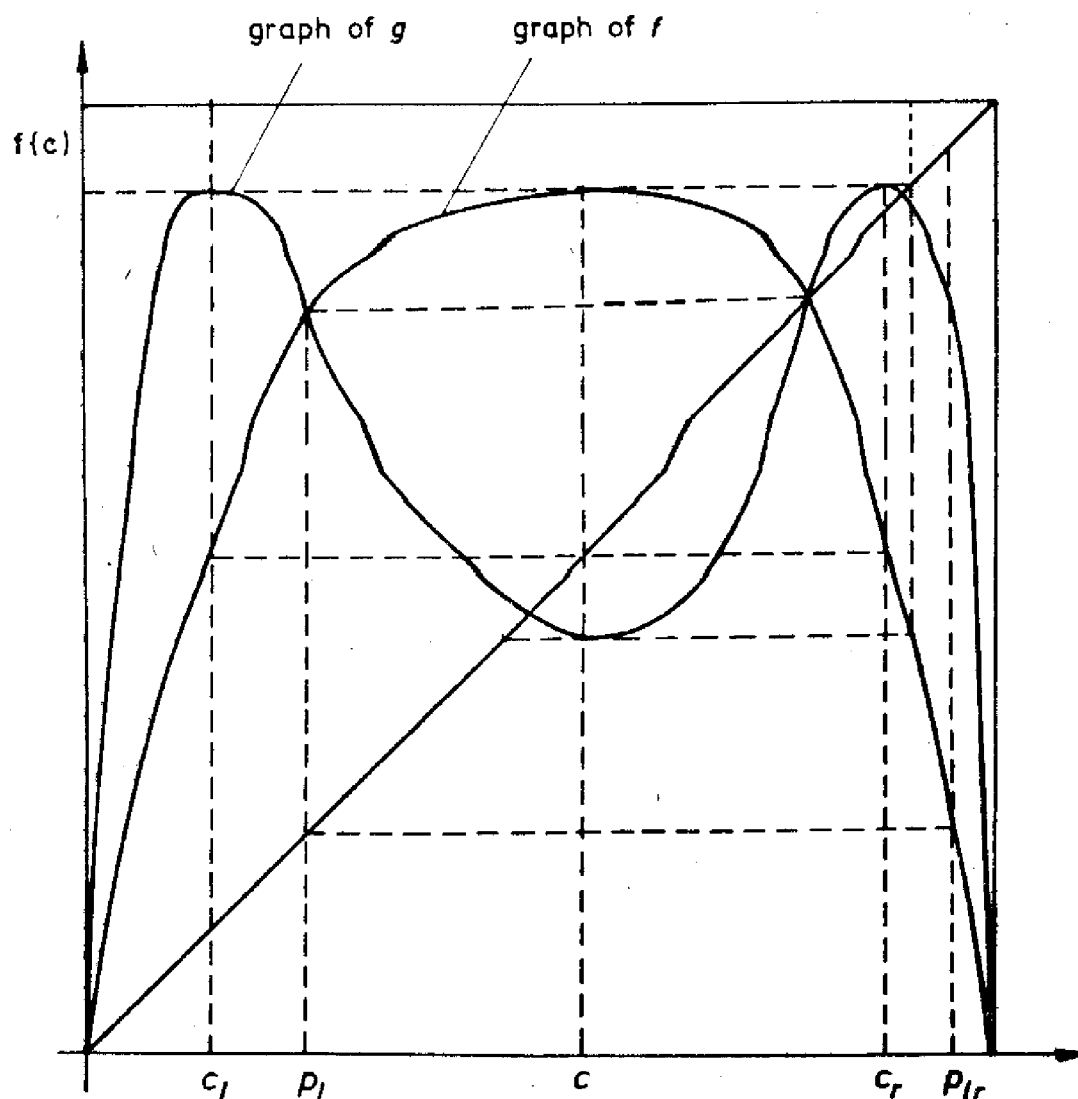
Proof of Theorem. We will inductively construct sequences $\{t_k\}, \{s_k\}$. We have $f_{t_0}^3(c_{t_0}) = 0 < c_{t_0}$. Proposition 2, Lemma 6 and continuity of F imply that for t from some non-empty interval $[s_0, t_0]$ f_t has 3-periodic point (by [4] n -periodic point for every n). Let $t_1 = \inf\{t < t_c: f_t^3(c_t) \leq p_t\}$. Lemma 5 implies that f_t has not $(2n+1)$ -periodic points for $t \in [t_c, t_1]$. Let us note now that the induction step requires to consider the family $\{f_{1t}\}$ given by Lemma 5 as $f_{1t} = (f_t)_1$ which for $t \leq t_1$ satisfies assumptions of the Theorem. The following questions remain still unsolved.

$$\text{Whether } \lim_{k \rightarrow \infty} s_k = \lim_{k \rightarrow \infty} t_k = t_c?$$

$$\text{Does (11) hold for } t \in [s_k, t_0]?$$

5. One can apply the above Theorem to the family of functions given by (1). Note additionally that the condition $f_t^3(c_t) \leq p_t$ is then equivalent to the inequality $f_t^2(c_t) \geq (p_t)_1$ which yields

$$t^4 - 4t^3 + 16 \geq 0,$$



which is satisfied for $t \in [t_1, 4]$ (approx. $t_1 = 3.6786$). So for $t \leq 3.6786\dots$, there are not $(2n+1)$ -periodic points. Moreover Proposition 2 implies that f_t has not 3-periodic points for $t < u$, where $u > t_1$ is some number.

The author wishes to thank Professor Stanisław Śędziwy for many helpful remarks.

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Received July 9, 1981.