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The Canonical Form of the Ricci Tensor in the n-dimensional Lorentz Space

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The algebraic classification of the Ricci tensor in general relativity has been discussed by several authors (Churchil 1932, Plebanski 1964, Plebanski and Stachel 1968, Barnes 1974).

G. S. Hall [2] gave an especially simple method of this classification. In this method G. S. Hall used the so-called null rotation subgroup of the proper ortochronous Lorentz group. A useful for computation form of this group is given in [2].

In this paper I present a generalization of Hall's method in the case of an n-dimensional Lorentz space. (In Hall's paper $n = 4$). I give a general classification of symmetric tensor of degree 2 in this space.

Let M be an n-dimensional Lorentz manifold with the metric tensor g. Hence g is a symmetric tensor field of the type $(0, 2)$, which is nondegenerate and has the same sygnature $(-, \underbrace{+, +, \dots, +}_{n-1 \text{ times}})$ at each point. Let $g_{\alpha\beta}$ be the component functions of the metric tensor g. We assume that indexes $\alpha, \beta, \gamma, \dots$ take values $1, 2, \dots, n$ and $p, r, s, t, \dots$ take values $1, 2, \dots, n-2$.

Let $T_x M$ be the tangent space at x and l be the null vector with components l^α ($l \neq 0$, $g(l, l) = 0$). According to Witt's theorem there exist vectors $k_1, k_2, \dots, k_{n-2}, m$ (with components $k_1^\alpha, k_2^\alpha, \dots, k_{n-2}^\alpha, m^\alpha$) such that $l, m, k_1, k_2, \dots, k_{n-2}$ form a basis for $T_x M$ and:

$$(1) \qquad \begin{aligned} & g(l, m) = g(k_s, k_s) = 1 \\ & g(l, k_s) = g(m, k_s) = g(m, m) = g(l, l) = g(k_s, k_t) = 0 \\ & \text{for } s \neq t. \end{aligned}$$

Now we define the subgroup G of the ortochronous Lorentz group under the following conditions:

a) transformations of G preserve metric conditions (1) (in general relativity ($n = 4$) we say that transformations of G preserve causal character of vectors).

b) for every $f \in G$ there exist such $b^1 > 0$ that $f(l) = b^1 \cdot l$.

The transformation of G has the following form:

(2)
$$\begin{aligned} l^\alpha \to l'^\alpha &= b^1\cdot l^\alpha \\ m^\alpha \to m'^\alpha &= b^2\cdot l^\alpha+(b^1)^{-1}\cdot m^\alpha+d^s\cdot k_s^\alpha \\ k_s^\alpha \to k_s'^\alpha &= b^{s+2}\cdot l^\alpha+a_s^t\cdot k_t^\alpha \end{aligned}$$

where: $b^1, \ldots, b^n, d^1, \ldots, d^{n-2}, a_s^t \in \boldsymbol{R}$ $(t, s = 1, \ldots, n-2)$,

$b^1>0$, the matrix $[a_s^t] \in \mathrm{SO}(n-2)$, $b^{s+2} = -b^1\cdot d^t\cdot a_t^s$,

$$b^2 = -\frac{b^1}{2}\sum_{s=1}^{n-2}(d^s)^2.$$

The matrix of transformation (2) in basis (1) has the form:

(3)
$$\begin{bmatrix} b^1 & b^2 & b^3 & \ldots & b^n \\ 0 & (b^1)^{-1} & 0 & \ldots & 0 \\ 0 & d^1 & a_1^1 & \ldots & a_{n-2}^1 \\ 0 & d^2 & a_1^2 & \ldots & a_{n-2}^2 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & d^{n-2} & a_1^{n-2} & \ldots & a_{n-2}^{n-2} \end{bmatrix}$$

Later on we shall also use the inverse matrix of (3) in the case when $a_s^t = \delta_s^t$:

$$\begin{bmatrix} (b^1)^{-1} & b^2 & d^1 & d^2 & \ldots & d^{n-2} \\ 0 & b^1 & 0 & 0 & \ldots & 0 \\ 0 & -d^1b^1 & 1 & 0 & \ldots & 0 \\ 0 & -d^2b^1 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & -d^{n-2}b^1 & 0 & \cdot & \ldots & 1 \end{bmatrix}$$

A covariant tensor of rank 2 (for example the Ricci tensor) can be presented in a (l, m, k_s)-basis in the form:

(5)
$$R_{\alpha\beta} = R^1\cdot l_\alpha\cdot l_\beta+2\cdot R^2\cdot l_{(\alpha}m_{\beta)}+R^3\cdot m_\alpha\cdot m_\beta+2\cdot T^s\cdot l_{(\alpha}k_{|s|\beta)} +2\cdot N^s\cdot m_{(\alpha}k_{|s|\beta)}-\sum_{s=1}^{n-2} S^{ss}k_{s\alpha}k_{s\beta}+2\cdot S^{st}\cdot k_{s(\alpha}k_{|t|\beta)}\,.$$

Lifting index β in $R_{\alpha\beta}$ we obtain a linear transformation R_α^β, which in the sequel will be called R. Applying (5) we will compute values of linear transformation R for basic vectors:

(6)
$$\begin{cases} R(l) = R^2\cdot l+R^3\cdot m+N^s\cdot k_s \\ R(m) = R^1\cdot l+R^2\cdot m+T^s\cdot k_s \\ R(k_t) = T^t\cdot l+N^t\cdot m+S^s\cdot k_s \end{cases}$$

Matrix R in a (l, m, k_s)-basis is of the form:

(7)
$$\begin{bmatrix} R^2 & R^1 & T^1 & T^2 & \dots & T^{n-2} \\ R^3 & R^2 & N^1 & N^2 & \dots & N^{n-2} \\ N^1 & T^1 & S^{11} & S^{21} & \dots & S^{n-2\ 1} \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ N^{n-2} & T^{n-2} & S^{1\,n-2} & S^{2\,n-2} & \dots & S^{n-2\ n-2} \end{bmatrix}$$

Now we will give some transformation laws for matrix (7) under the change of basis (2) in situation when $a_t^s = \delta_t^s$ and with condition (7) $S^{st} = 0$ for $s \neq t$ fulfilled:

(8.1) $$\bar{R}^1 = R^1 \cdot (b^1)^{-2} + 2 \cdot R^2 \cdot b^2 \cdot (b^1)^{-1} + R^3 \cdot (b^2)^2 + 2 \cdot b^2 \sum_{s=1}^{n-2} N^s \cdot d^s + 2 \cdot (b^1)^{-1} \sum_{s=1}^{n-2} T^s \cdot d^s + \sum_{s=1}^{n-2} S^{ss} \cdot d^s$$

(8.2) $$\bar{R}^2 = R^2 + R^3 \cdot b^1 \cdot b^2 + b^1 \cdot \sum_{s=1}^{n-2} N^s \cdot d^s$$

(8.3) $$\bar{R}^3 = R^3 \cdot (b^1)^2$$

(8.4) $$\bar{N}^r = N^r \cdot b^1 - R^3 \cdot d^r \cdot (b^1)^2$$

(8.5) $$\bar{T}^r = -R^3 \cdot d^r \cdot b^1 \cdot b^2 + N^r \cdot d^r - R^2 \cdot d^r + T^r \cdot (b^1)^{-1} + S^{rr} d^r - \sum_{s=1}^{n-2} N^s d^{s2} b^1 .$$

(Formula: 8.2, 8.3, 8.4 is true for $S^{st} = 0$ for $s = t$ too.)

In the sequel we will use the well known theorem:

LEMMA 1. *Every linear transformation of $T_x M$ has a 2-dimensional invariant subspace.*

A k-dimensional ($k \geqslant 2$) subspace of $T_x M$ is called:

a) *timelike*, if it contains two independent null vectors.

b) *lightlike*, if it contains exactly one null direction.

c) *spacelike*, if it does not contain the null vector.

LEMMA 2. *If R has a spacelike $n-2$ dimensional invariant subspace V, the subspace $V^\perp$ is invariant too.*

Proof. If V is a spacelike $n-2$ dimensional invariant subspace of R we can construct such a (l, m, k_s)-basis that vectors k_s span this subspace V. From condition (6) we get:

$$N^s = T^s = 0$$

The two first formulas of condition (6) show that $V^\perp$ is invariant.

LEMMA 3. *If R has a spacelike $n-2$ dimensional invariant subspace, R has $n-2$ orthogonal spacelike eigenvectors.*

Proof. We will use the well-known fact that a symmetric matrix can be presented in the diagonal by applying an orthogonal transformation. We take a (l, m, k_s)-basis in which vectors k_s span the invariant spacelike subspace.

Applying Lemma 1 we get $N^s = T^s = 0$. Matrix $[S^{st}]$ is symmetric (dimension $(n-2)\times(n-2)$). For matrix $[S^{st}]$ we take the matrix $[a_s^t] \in SO(n-2)$ which changes vectors k_s into k'_s in order to get the new, diagonal matrix $[S'^{st}]$. The transformation of type (2) with $b^1 = 1$, $d^1 = \ldots = d^{n-2} = 0$ and the above a_s^t show that $n-2$ orthogonal spacelike vectors do exist.

LEMMA 4. *A linear transformation R has a lightlike 2-dimensional invariant subspace if and only if R has a null eigenvector.*

Proof. If R has a lightlike 2-dimensional invariant subspace, we take the (l, m, k_s)-basis in which vectors l, k_1 span this subspace. In condition (6) we get:

$$R^3 = N^2 = \ldots = N^{n-2} = 0 \text{ (in the first equation)} \tag{9}$$

$$N^1 = 0 \qquad \text{(in the third equation).} \tag{10}$$

Condition (9) and (10) show that l is an eigenvector. We apply the first equation of condition (6).

Now we shall show, that if R has a null eigenvactor, it has a 2-dimensional lightlike subspace. Let l be the null eigenvactor. In Lemma 1 R has a 2-dimensional invariant subspace V. We have the following cases: V is spacelike, timelike or lightlike. If V is spacelike, we show that R has two spacelike eigenvectors. Any of these spacelike vectors and vector l span a lightlike invariant subspace. Now, if V is timelike, we take a (l, m, k_s)-basis in which l and m span the subspace V. In (6) we have:

$$T^s = N^s = 0 \quad \text{for every } s.$$

These condition show, that R has an $n-2$ dimensional spacelike invariant subspace. In Lemma 3 we show that R has $n-2$ spacelike eigenvectors. One of these eigenvectors and the vector l span a 2-dimensional lightlike invariant subspace. This finishes the proof.

LEMMA 5. *If R has a timelike 2-dimensional invariant subspace, it has $n-2$ spacelike eigenvectors.*

Proof. Let V be the 2-dimensional invariant subspace. By (6), $V^\perp$ is invariant too. $V^\perp$ is spacelike and according to Lemma 3 it has $n-2$ eigenvectors.

LEMMA 6. *Let $n = 3$. If R has no null eigenvector, it has a spacelike eigenvector.*

Proof. Let u be an eigenvector for R. A causal character of u is timelike or spacelike. If u is timelike, easy computations show that spacelike vectors also exist (according to the symmetry of R).

Theorem. *If the transformation R of the n-dimensional Lorentz space ($n \geqslant 3$) has no null eigenvector and has a 2-dimensional spacelike invariant subspace V*

1) *R has an $n-2$ dimensional invariant subspace $\tilde{V}$*

2) *R has $n-2$ spacelike eigenvectors*

3) *subspace $\tilde{V}^{\perp}$ is invariant too.*

Proof. If condition 1) is true, by applying Lemmas 3 and 2, we get conditions 2) and 3). We can prove these conditions by the means of induction. When $n = 3$, the assertion is included in Lemma 6. When $n = 4$ this theorem is trivial. Now we assume that the theorem is true for dimension $n-2$. In an n-dimensional Lorentz space we take a (l, m, k_s)-basis in which vector k_{n-3} and k_{n-2} span the subspace V. By condition 6 we get:

$$S^{n-3\,t} = S^{n-2\,s} = 0 \text{ for } t \neq n-3 \text{ and } s \neq n-2$$

$$T^{n-2} = T^{n-3} = N^{n-2} = N^{n-3} = 0$$

This fact implies $V^{\perp}$ is invariant. Let F get restriction of R to $V^{\perp}$. F is a linear transformation of the $n-2$ dimensional Lorentz space. $F_{\alpha\beta}$ is symmetric and F has 2-dimensional timelike or spacelike invariant subspace. In fact, when Lemma 5 is applied, F has a 2-dimensional spacelike invariant subspace. Thanks to induction this theorem is true for any n.

The classification. Here, like G. S. Hall's paper, two cases are considered:

1) *R has a null eigenvector.*

We take a (l, m, k_s)-basis in which l is the null eigenvector. Applying condition (6) we receive: $R^3 = N^1 = \ldots = N^{n-2} = 0$. We use transformation (2) with $b^1 = 1$, $b^2 = \ldots = b^n = 0$, $d^1 = \ldots = d^{n-2} = 0$ and a_s^t in which $S^{st} = 0$ for $s \neq t$.

Now we consider two subcases:

1a) *In matrix (7) $S^{ss} \neq R^2$ for every s:*

We apply the transformation of type (2) with $b^1 = 1$ and $d^1, d^2, \ldots, d^{n-2}$ which we obtain by equations:

$$\bar{T}^r = T^r \cdot (b^1)^{-1} - R^2 \cdot d^r + S^{\cdot rr} d^r = 0$$

and (8.5) with $R^3 = N^1 = \ldots = N^{n-2} = 0$. We finally get:

$$(11)\qquad \begin{bmatrix} R^2 & R^1 & 0 & 0 & \ldots & 0 \\ 0 & R^2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & S^{11} & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \ldots & S^{n-2\,n-2} \end{bmatrix}$$

1b) *There exist indeces $r_1, \ldots, r_w$ such that in matrix (7) $S^{r_1 r_1} = \ldots = S^{r_w r_w} = R^2$.*

Like in subcase 1a we get finally:

(12)
$$\begin{bmatrix} R^2 & 0 & T^1 & \dots & T^w & 0 & \dots & 0 \\ 0 & R^2 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & T^1 & R^2 & \dots & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ 0 & T^w & 0 & \dots & R^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & S^{w+1\,w+1} & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & S^{n-2\,n-2} \end{bmatrix}$$

2) *R has no null eigenvector*:
In this case we apply the theorem and we get:

(13)
$$\begin{bmatrix} R^2 & R^1 & 0 & \dots & 0 \\ R^3 & R^2 & 0 & \dots & 0 \\ 0 & 0 & S^{11} & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & S^{n-2\,n-2} \end{bmatrix}$$

My classification for dimension $n = 4$ is in accordance with the G. S. Hall's classification.

References

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