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On the Existence of Solution of Homogeneous Boundary Value Problem for Ordinary Differential Equations

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Denote by C^n the space of all continuous n-dimensional vector functions $\xi(t) = (\xi_1(t), \ldots, \xi_n(t))$ on $[0, h]$ with the norm $|\xi(t)| = \sum_{i=1}^{n} \max\{|\xi_i(t)|: t \in [0, h]\}$.

Let R^n be the n-dimensional Euclidean space. Consider a system of differential equations

$$x'(t) = f(t, x(t)) + g(t, x(t)), \tag{1}$$

where $x(t) = (x_1(t), \ldots, x_n(t))$ is an element of C^n space and

$$f(t, x) = (f_1(t, x_1, \ldots, x_n), \ldots, f_n(t, x_1, \ldots, x_n)),$$

$$g(t, x) = (g_1(t, x_1, \ldots, x_n), \ldots, g_n(t, x_1, \ldots, x_n))$$

are n-dimensional real vector functions defined on $[0, h] \times R^n$.

Assume that the functions $f_i(t, x_1, \ldots, x_n)$, $g_i(t, x_1, \ldots, x_n)$ are continuous and $f(t, x)$ is homogeneous with respect to x, which means that for every $x \in R^n$ and real number λ the condition $f(t, \lambda x) = \lambda f(t, x)$ is satisfied.

Suppose that

$$|g(t, x)| \leqslant \varphi(t) + \varepsilon |x|,$$

where $\varphi(t)$ is a summable function in the interval $[0, h]$ and $\varepsilon \geqslant 0$ is a constant. Let L be a continuous, homogeneous mapping of C^n into R^n.

THEOREM. *If the function identically equal to zero is the only solution of equation*

$$x'(t) = f(t, x(t)) \tag{2}$$

with the condition

$$Lx = 0, \tag{3}$$

then there exists a constant $k > 0$ such that for $\varepsilon < k$ and for every $r \in R^n$ there exists at least one solution of system (1) *satisfying the condition*

$$Lx = r. \tag{4}$$

Proof. The solution of problem (2), (3) is equivalent to the solution of equation

$$x(t) = \int_0^t f(s, x(s))ds + Lx(t) - x(0) \tag{5}$$

and the solution of problem (1), (4) is equivalent to the solution of equation

$$x(t) = \int_0^t f(s, x(s))ds + \int_0^t g(s, x(s))ds + Lx(t) - x(0) - r. \tag{6}$$

Denote by A a mapping of C^n into C^n such that

$$x(t) \to \int_0^t f(s, x(s))ds + Lx(t) - x(0)$$

and by b a mapping of C^n into C^n such that

$$x(t) \to \int_0^t g(s, x(s))ds - r.$$

The equation (5) can be rewritten in the form

$$x = Ax \tag{7}$$

and (6) can be written as

$$x = Ax + bx. \tag{8}$$

Consider the equation

$$x = Ax + \lambda bx \tag{9}$$

where λ is a constant, $0 \leqslant \lambda \leqslant 1$.

Denote by $S(\lambda)$ the set of all solutions of equation (9). The mapping

$$x \to Ax + \lambda bx$$

is completely continuous. Hence, from the Leray-Schauder alternative [1], it follows that in order to prove the existence of solution of equation (8) it is sufficient to show that the set $S(\lambda)$ is bounded by some constant independent of the parameter λ.

First we show that for $u, v \in C^n$ satisfying the equation

$$u = Au + v \tag{10}$$

there exists a constant η such that $|u| \leqslant \eta |v|$.

Suppose the contrary. Then for every natural number n there exists x_n, y_n in C^n such that

$$|u_n| > n|v_n|$$

Setting $w_n = \frac{u_n}{|u_n|}$ we have

$$|w_n| = 1 \text{ and}$$

$$w_n = Aw_n + \frac{v_n}{|u_n|}. \tag{11}$$

By the complete continuity of the mapping A there exists a subsequence $\{w_{\alpha_n}\}$ convergent to a point, say z^*.

Passing in (1) to the suitable subsequences we have at limit

$$z^* = Az^*.$$

Since $|z^*| = 1$, we have the contradiction with the assumption that the unique solution of equation (7) is identically equal to zero.

Let $x \in S(\lambda)$. Setting $u = x$, $v = \lambda bx$, we get

$$|x| \leqslant \eta\lambda|bx| \leqslant \eta\Big(\int_0^t \varphi(s)ds + \varepsilon\int_0^t |x|ds + |r|\Big).$$

Hence, we obtain

$$|x| \leqslant \frac{\eta\left(\int_0^t \varphi(s)ds + |r|\right)}{1-\eta\varepsilon h}. \tag{12}$$

This inequality can be fulfilled only if

$$\varepsilon < \frac{1}{\eta h}.$$

If we set $k = \frac{1}{\eta h}$, then, by (12), the set $S(\lambda)$ is bounded.

This completes the proof of Theorem.

Remark. In order to prove our Theorem we can also apply the Theorem 2.1 [2].

References

[1] A. Granas, *The theory of compact vector fields and some of its applications to topology of functional spaces*, Rozprawy Matematyczne, PAN, 30 (1962).

[2] A. Lasota, *Une généralisation du premier théorème de Fredholm et ses applications a la théorie des équations différentielles ordinaires*, Ann. Polon. Math. 18 (1966), pp. 65—77.

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