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# Periodic and Constant Solutions Via Topological Principle of Ważewski

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**Introduction.** The purpose of this paper is to show what conditions concerning differential equations or dynamical systems one need to add to assumptions of Ważewski's topological principle to obtain the existence of rest points or periodic trajectories. We will use this principle as stated in [3], the formulation somewhat different from the original one presented in [9].

In Chapter 1 we introduce the notion of the semidynamical system and the periodic process and we present Ważewski's Theorem. The basic result of this paper is Theorem 1 in Chapter 2. It says that if $p$ is a $T$-periodic process on a topological space $X$, such that its trajectories escape from a set $P \subset X$ only by $P^- \subset \partial P$ and we assume that $P$ and $P^-$ are compact ANR-spaces and difference between their Euler characteristics is nonzero, than there exists a $T$-periodic trajectory of $p$ staying in the set $P$. In the remaining chapters of this paper we apply this theorem to differential equations. We formulate theorems in which we obtain the existence of periodic solutions in a set $B$ which is the intersection of $\{L^k \leqslant 0\}$ for some real valued functions $L^k$, $k = 1, \ldots, m$. The assumptions of these theorems are similar to those in [6] pp. 280—283. As a special result we obtain theorems about the existence of zeros of a vector-field in a given set $B$. In Chapter 4 and 5 we introduce and apply the technique of isolating blocks and isolated invariant sets which was developed mostly by C. Conley and R. Churchill.

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**1. Preliminaries.** Let $X$ be a metrizable space. We say that

$$\pi\colon [0, \infty) \times X \to X$$

is a semidynamical system on $X$ if $\pi$ is continuous, $\pi(0, x) = x$ and $\pi(s+t, x) = \pi(t, \pi(s, x))$ for any $x \in X$ and $s, t \in [0, \infty)$.

By a process on $X$ we understand a continuous mapping

$$p\colon [0,\infty)\times X\times \boldsymbol{R}\ni(\tau,x,t)\to p(\tau,x,t)\in X$$

such that the mapping

$$\pi\colon [0,\infty)\times(X\times\boldsymbol{R})\ni\big(\tau,(x,t)\big)\to\big(p(\tau,x,t),t+\tau\big)\in X\times\boldsymbol{R}$$

is a semidynamical system. This $\pi$ is a called the corresponding system to $p$.

The notions of a semidynamical system and a process arise in a natural way in the theory of differential equations. For later purposes we shall describe it briefly (see [6] for details).

Let $\Omega\subset\boldsymbol{R}^n$ be an open set, let

$$f\colon \boldsymbol{R}\times\Omega\to\boldsymbol{R}^n$$

be continuous and suppose for any $(t_0,x_0)\in\boldsymbol{R}\times\Omega$ the initial value problem

$$\text{(i)}\qquad \dot{x}=f(t,x),\ x(t_0)=x_0$$

has unique solution

$$\text{(ii)}\qquad x=\varphi(\tau;\ t_0,x_0)$$

defined for all $\tau\geqslant t_0$.

The function $p$ described by

$$p(\tau,x,t)=\varphi(\tau-t;\ t,x)$$

is a process. If $f$ doesn't depend on $t$, the mapping $\pi$

$$\pi(t,x)=\varphi(t;\,0,x)$$

is a semidynamical system. If $f$ is $T$-periodic in $t$, then the corresponding process satisfies

$$p(\tau,x,t)=p(\tau,x,t+T)$$

for all $x$ and $t$. Such a process will be called $T$-periodic.

For a set $W\subset X$, define the subsets $W^-$ and $W^0$ by:

$$W^-=\{x\in W\colon \exists\varepsilon_n>0,\ \varepsilon_n\to 0,\ \pi(\varepsilon_n,x)\notin W\}$$

$$W^0=\{x\in W\colon \exists t>0,\ \pi(t,x)\notin W\}$$

The set $W$ is called the Ważewski set if it satisfies the following conditions:

$$\text{(W1)}\qquad x\in W,\ \pi([0,t],x)\subset\overline{W}\Rightarrow\pi([0,t],x)\subset W$$

$$\text{(W2)}\qquad W^-\ \text{is closed relative to}\ W^0.$$

**Theorem** 0 (see [3] or [4], p. 24). *If $W$ is a Ważewski set then the function*:

$$\tau\colon W^0\ni x\to\sup\{t\geqslant 0\colon \pi([0,t],x)\subset W\}\in\boldsymbol{R}$$

*is continuous.*

Let $K$ be a compact ANR i.e. absolute neighbourhood retract (for the definition and properties see [1], chapter V). By $\chi(K)$ we denote the Euler characteristics of $K$.

$$\chi(K) = \sum_{q=0}^{\infty}(-1)^q \dim H_q(K)$$

Here $H = \{H_q\}_{q\in Z}$ is the singular homology functor with coefficients in the field $\boldsymbol{Q}$ of rational numbers.

**2. The basic results.** The basic result of this paper is the following:

THEOREM 1. *Let $p$ be a $T$-periodic process on $X$. Let $P$ and $P^-$, $P^- \subset P$, be subsets of $X$ such that*

(A1) *$P$ and $P^-$ are compact* ANR-*s*

(A2) $P^- \times \boldsymbol{R} = \{(x, t) \in P\times \boldsymbol{R}\colon \exists\, \varepsilon_n > 0,\ \varepsilon_n \to 0,\ p(\varepsilon_n, x, t) \notin P\}$

(A3) $\chi(P) - \chi(P^-) \neq 0$

*Then there exists $x_0 \in P$ such that $p(\tau, x_0, 0) \in P$ and*

$$p(\tau, x_0, 0) = p(\tau + T, x_0, 0)$$

*for any $\tau \geqslant 0$.*

Proof. The periodicity of the process $p$ implies that it suffices to prove that $x_0 = p(T, x_0, 0)$ for some $x_0 \in P$. Let $\pi$ be the semidynamical system on $X\times \boldsymbol{R}$ corresponding to $p$ and let $W = P\times \boldsymbol{R}$. By (A2), $W^- = P^-\times \boldsymbol{R}$. It is easy to check that $W$ is the Ważewski set for $\pi$. Hence, by Theorem 0 the function

$$c\colon P \ni x \to \min\{T, \sup\{\tau \in [0, \infty)\colon p(0, \tau], x, 0) \subset P\} \in \boldsymbol{R}$$

is continuous.

Let $P\oplus(P^-\times S^1)$ denote the topological direct sum of $P$ and $P^-\times S^1$ where $S^1$ is the unit circle in a complex plane. Identifying in $P\oplus(P^-\times S^1)$ points $y\in P^-$ and $(y, 1) \in P^-\times S^1$ we obtain a new space $Z$ with the quotient topology. Let

$$q\colon P\oplus(P^-\times S^1) \to Z$$

be the quotient mapping. We introduce two functions

$$\Phi\colon Z\times[0, 1] \to Z \quad \text{and} \quad \varphi\colon Z \to Z$$

by the formulae

$$\Phi(z, \tau) = \begin{cases} q(p(\tau T, x, 0)), & \text{for } z = q(x),\ x\in P,\ \tau \leqslant \dfrac{c(x)}{T} \\ q\left(p(c(x), x, 0), \exp\left(\pi i\left(\tau - \dfrac{c(x)}{T}\right)\right)\right), & \text{for } z = q(x),\ x\in P,\ \tau \geqslant \dfrac{c(x)}{T} \\ q((x, \alpha\exp(\pi i\tau), & \text{for } z = q(x, \alpha),\ x \in P^-,\ \alpha\in S^1 \end{cases}$$

$\varphi(z) = \Phi(z, 1)$.

The continuity of $c$ implies that $\Phi$ and $\varphi$ are continuous. Let $A = q(P)$, $B = q(P^{-} \times S^{1})$. Observe that $A$ is homeomorphic to $P$, $B$ to $P^{-} \times S^{1}$ and $A \cap B$ to $P^{-}$. By Theorem V.2.9 in [1] $Z$ is a compact ANR. We have:

$$\chi(Z)+\chi(A \cap B) = \chi(A)+\chi(B)$$
$$H_q(P^{-} \times S^{1}) = H_{q-1}(P^{-}) \oplus H_q(P^{-})$$

(see [5] for the proofs). By (A3)

$$\chi(Z) = \chi(P)-\chi(P^{-}) \neq 0$$

The mapping $\Phi$ is a homotopy joining $\varphi$ to the identity on $Z$. Thus the Lefschetz number of $\varphi$ is equal $\chi(Z)$, and by the Lefschetz-Hopf Fixed Point Theorem (see [1] or [5]), $\varphi$ has a fixed point $z_0 \in Z$. We see that $z_0 = q(x_0)$ for some $x_0 \in P$ and $x_0 = p(T, x_0, 0)$.

COROLLARY 1. *Let $\pi$ be a semidynamical system on $X$. Let $W \subset X$ and $W^{-}$ are compact* ANR-*s. If $\chi(W)-\chi(W^{-}) \neq 0$, then there exists $x_0 \in W$ such that for any $t \geqslant 0, \pi(t, x_0) = x_0$.*

Proof. For arbitrary $T$ we may consider $\pi$ as a $T$-periodic process. Set $T = 2^{-n}$, $n \in N$. By Theorem 1 there exists a sequence $\{x_n\}$ of points satisfying $x_n = \pi(2^{-n}, x_n)$. If $x_0$ is a limit points of $\{x_n\}$ then $\pi(t, x_n) = x_0$ for $t \geqslant 0$.

**3. Applications to differential equations.** In the sequel we shall consider the differential equation (i) with $\Omega$ and $f$ defined as in Chapter 1.

Let $p$ and $q$ be nonnegative integers, $p+q > 0$. Let functions $L^{k}$, $k = 1, \dots, p+q$, be of class $C^{1}$ in $\Omega$. Define the sets:

$$B = \overline{\{x \in \Omega\colon L^{x}(x) < 0,\ k = 1, \dots, p+q\}}$$

(closure relative to $\Omega$),

$$\pi^{j} = \{x \in B\colon L^{j}(x) = 0\}$$

for $j = 1, \dots, p+q$. The set $B$ defined above will be called a block of type $(p, q)$ described by the functions $\{L^{k}\}_{k=1,\dots,p+q}$ if the following conditions are satisfied:

(B1) $\quad \nabla L^{k}(x)\cdot f(t, x) > 0 \quad$ for $(t, x) \in \boldsymbol{R} \times \pi^{k} \quad k = 1, \dots, p$

(B2) $\quad \nabla L^{k}(x)\cdot f(t, x) < 0 \quad$ for $(t, x) \in \boldsymbol{R} \in \pi^{k} \quad k = p+1, \dots, p+q$

($\nabla L$ denotes the gradient of $L$ and the dot denotes the scalar product). Let

$$b^{-} = \bigcup_{k=1}^{p} \pi^{k}$$

THEOREM 2. *Assume that $f\colon \boldsymbol{R} \times \Omega \to \boldsymbol{R}^{n}$ is of class $C^{1}$ and T-periodic in t. Let B, defined as above, be a block of the type $(p, q)$. If $B$ and $b^{-}$ are compact ANR-s and $\chi(B)-\chi(b^{-}) \neq 0$, then there exists a point $x_0 \in \operatorname{int} B$ such that the solution $\varphi(\tau, 0, x_0)$ of the problem* (i) *is T-periodic in $\tau$. Moreover, $\varphi(\boldsymbol{R}, 0, x_0) \subset \operatorname{int} B$.*

Proof. Without the loss of generality we can assume that all solutions of (i) exist for all $\tau \in \boldsymbol{R}$. For this purpose modify, if necessary, $f$ outside an open set $V$ containing $B$ by setting $\tilde{f}(t, x) = g(x)f(t, x)$, where $g\colon \Omega \to [0, 1]$ is of class $C^1$, $g(x) = 0$ for $x \notin V$ and $g(x) = 1$ for $x \in B$. Clearly, the solutions of the equation $\dot{x} = \tilde{f}(t, x)$ satisfying the assertion of the theorem coincide with the solutions of (i). Thus (i) defines a $T$-periodic process $p$ as described in Chapter 1. The assumptions imply (A1), (A2) and (A3) hence the assertion follows from Theorem 1.

Remark 1. In Theorem 2 it is assumed that $B$ and $b^-$ are ANR-$s$. This is true if, in particular, functions $\{L^k\}$ satisfy the following condition:

(N) For any $x \in \Omega$ either the vectors $\nabla L^k(x)$, $k \in K(x) = \{j\colon L^j(x) = 0\}$ are lineary independent, or the set $K(x)$ is empty.

In a fact, by the Inverse Function Theorem, $B$ and $b^-$ are locally ANR-$s$. The Second Theorem of Hanner (see [1]) implies that they are ANR-$s$.

In the case of autonomous differential equations we can prove the following:

THEOREM 3. *Let $f\colon \Omega \to \boldsymbol{R}^n$ be continuous. Let be a block of type $(p, q)$ described by $\{L^k\}$. Assume that instead of the strong inequalities* (B1) *and* (B2) *there are weak inequalities fulfilled*:

$$\text{(C1)} \qquad \nabla L^k(x)\cdot f(x) \geqslant 0 \quad \textit{for } x \in \pi^k, \quad k = 1, \ldots, p$$

$$\text{(C2)} \qquad \nabla L^k(x)\cdot f(x) \leqslant 0 \quad \textit{for } x \in \pi^k, \quad k = p+1, \ldots, p+q$$

*If $\{L^k\}$ satisfy* (N) *and $\chi(B) - \chi(b^-) \neq 0$, then there exist $x_0 \in B$ such that $f(x_0) = 0$.*

Proof. From (N) and results of [7] it follows that the function $f$ can be uniformly approximated by $C^1$-functions satisfying (D1), (D2):

$$\text{(D1)} \qquad \nabla L^k(x)\cdot f(x) > o \text{ for } x \in \pi^k,\ k = 1, \ldots, p$$

$$\text{(D2)} \qquad \nabla L^k(x)\cdot f(x) < 0 \text{ for } x \in \pi^k,\ k = p+1, \ldots, p+q.$$

By applying Corollary 1 to these functions we obtain a set of points from B. Any limit point of this set fulfils the thesis.

**4. Isolated invariant sets.** In the sequel we assume that $\Omega \subset \boldsymbol{R}^n$ is open and $C^1$-function $f\colon \Omega \to \boldsymbol{R}^n$ induces a dynamical system $\pi$ on $\Omega$.

We say that a compact set $K \subset \Omega$ is an isolated invariant set for the system $\pi$, if there exists an open neighbourhood $U$ of $K$ in $\Omega$, called an isolating neighbourhood, such that $K$ is the maximal invariant set relative to $\pi$ in $U$.

Let $B$ be a block of type $(p, q)$ described by $\{L^k\}$. Since $f$ is $t$-independent, the conditions (B1), (B2) reduce to (D1), (D2). Observe that $K(B) = \{x \in B\colon \pi(\boldsymbol{R}, x) \subset B\}$ is an isolated invariant set ($\operatorname{int} B$ is its isolating neighbourhood).

We say that $B$ is an isolating block for $K$ if $K = K(B)$. Theorem 2.4 in [10] implies that for any neighbourhood $U$ of $K$ there exist a compact isolating block $B$ for $K$ of type (1,1) described by $C^\infty$-functions $L^1, L^2\colon \Omega \to \boldsymbol{R}$ such that $B$ is a manifold with corners, $b^-$ is a manifold with boundary and $B \subset U$. The elementary properties of ANR-*s* (see [1]) imply that B and $b^-$ are compact ANR-*s*.

In [2], Chapter 4 some cohomological properties of blocs are proved. Our definition of an isolating block differs slightly from that in [2], but little modifications of proofs of the mentioned paper permit us to assert that these properties are also valid in our situation. We present it here.

Let $\{\check{H}^k\}_{q\in Z} = \check{H}^*$ denote the Čech cohomology functor with coefficients in $Q$. There exists homomorphisms

$$\alpha^* = \{\alpha^q\}_{q\in Z},\ \beta^* = \{\beta^q\}_{q\in Z},\ \gamma^* = \{\gamma^q)_{q\in Z}$$

such that the following sequence (E) is exact:

$$(E)\ \dots \xrightarrow{\gamma^{q-1}} \check{H}^q(B, b^-) \xrightarrow{\alpha^q} \check{H}^q(K(B)) \xrightarrow{\beta^q} \check{H}^q(a^-) \xrightarrow{\gamma^q} \check{H}^{q+1}(B, b^-) \xrightarrow{\alpha^{q+1}} \dots$$

where $a^- = \{x \in b^-\colon \pi((-\infty, 0], x) \subset B\}$.

If $B_1$ and $B_2$ are blocks such that $K(B_1) = K(B_2)$, then

$$\check{H}^*(B_1, b_1^-) \cong \check{H}^*(B_2, b_2^-)$$

which in turn implies that if $K$ is an isolated invariant set, then $\check{H}^*(B, b^-)$ is independent (modulo isomorphisms) of a choice of an isolating block $B$ for $K$.

At the beginning of this chapter we showed that there exists $B_0$, an isolating block for $K$, such that $B_0$ and $b_0^-$ are compact ANR-*s*. Since coefficients of functors $\check{H}^*$ and $H$ are in $Q$, we have

$$\check{H}^*(B_0, b_0^-) \cong H(B_0, b_0^-).$$

Set

$$\check{\chi}(X, Y) = \sum_{q=0}^{\infty} (-1)^q \dim \check{H}^q(X, Y)$$

for a topological pair $X$, $Y$ such that $\check{H}^q(,X\ Y)$ is trivial for almost all $q$ and for any $q$ $\dim \check{H}^q(X, Y) < \infty$. Thus for the isolated invariant set $K$ the number $\check{\chi}(B, b^-)$ is well defined and independent of the choice of an isolating block $B$ (since

$$\check{\chi}(B, b^-) = \check{\chi}(B_0, b_0^-) = \chi(B_0, b_0^-)).$$

The number $\check{\chi}(B, b^-)$ is called the index of the isolated invariant set $K = K(B)$ and is denoted ind $K$.

**5. Results concerning isolated invariant sets.** The main result of this chapter is the following:

THEOREM 4. *If $K$ is an isolated invariant set for the dynamical system $\pi$ induced by $f$ and* $\operatorname{ind} K \neq 0$, *then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any function*

$$h\colon \boldsymbol{R} \times \Omega \to \boldsymbol{R}^n$$

*of class* $C^1$ *such that* $t \to h(t,x)$ *is T-periodic and* $|h(t,x)| < \delta$ *for all* $(t,x) \in \mathbf{R} \times \Omega$, *the equation*

$$\dot{x} = f(x) + h(t,x)$$

*has T-periodic solution* $\varphi$ *such that* $\operatorname{dist}\big(\varphi(\mathbf{R}), K\big) < \varepsilon$.

Proof. There exists an isolating block $B$ for $K$ contained in the $\varepsilon$-neighbourhood of $K$ such that $B$ and $b^-$ are compact ANR-*s* (see Chapter 4). Since we have the strong inequalities (D1), (D2) for this block, a perturbated function $(t,x) \to f(x) + h(t,x)$ fulfils (B1), (B2) for any small perturbation $h$. $\chi(B, b^-) \neq 0$ since $\operatorname{ind} K \neq 0$. The exact homology sequence of the pair $(B, b^-)$ implies that

$$\chi(B, b^-) = \chi(B) - \chi(b^-)$$

Now the thesis is a consequence of Theorem 2.

Corollary 2. *If* $f;\ \Omega \to K^n$ *is of class* $C^1$ *and* $B$ *is a block of type* $(p,q)$ *described by* $\{L^k\}$, *then* $\check{\chi}(B, b^-)$ *is well defined and if* $\check{\chi}(B, b^-) \neq 0$ *then the thesis of Theorem* 4 *is true. In particular there exists* $x \in B$ *such that* $f(x) = 0$. (In the latter case it suffices to assume continuity of $f$.)

Corollary 3. *If* $K$ *is asymptotically stable and* $\check{\chi}(K) \neq 0$, *then the thesis of Theorem* 4 *is true.*

Proof. This is a consequence of the exactness of (E). If $K$ is asymptotically stable, then $a^- = \emptyset$ and thus $\operatorname{ind} K = \check{\chi}(K) \neq 0$.

Corollary 4. *If* $n = 2$ *and* $K$ *is an isolated invariant set such that* $\dim \check{H}^0(K) < \dim \check{H}^1(K) < \infty$, *then Theorem* 4 *holds.* (The set "$\infty$" satisfies the assumption of this corollary.)

Proof. Let $B$ be an isolating block contained in the $\varepsilon$-neighbourhood of $K$ such that $b^-$ is a manifold with boundary. The exactness of (E) implies that

$$\check{\chi}(K) - \check{\chi}(a^-) = \check{\chi}(B, b^-).$$

The set $a^-$ is a compact subset of $b^-$. Since $\dim b^- = 1$, $b^-$ is a union of a finite number of circles $S^1$ and unit intervals $[0,1]$. Let $a^- = \bigcup_{i=1}^{m} a_i$ and $a_i$ are in the different components of $b^-$, $i = 1, \ldots m$. Then

$$\check{\chi}(a^-) = \sum \check{\chi}(a_i).$$

If $a_i = S^1$ then $\check{\chi}(a_i) = 0$. On the other hand $a_i$ is homeomorphic to a compact subset of $\mathbf{R}$ and thus $\check{\chi}(a_i) = \dim \check{H}^0(a_i)$. Hence

$$\check{\chi}(K) - \check{\chi}(a^-) = \dim \check{H}^0(K) - \dim \check{H}^1(K) - \sum \check{\chi}(a_i) < 0,$$

which implies that $\operatorname{ind} K \neq 0$.

**6. Examples.** Finally we give two examples concerning the theorems of this paper.

Example 1. The origin is the isolated invariant set for the dynamical system induced by the equation $\dot{x} = Ax$, where $A\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a linear map having eigenvalues with non-zero real parts. If $A$ has exactly $k$ eigenvalues with real parts positive, then $\operatorname{ind}\{0\} = (-1)^k$. Theorem 4 implies that if a function $h;\ \boldsymbol{R}^{n+1} \to \boldsymbol{R}^n$ is $T$-periodic for $t$ and bounded by sufficiently small constant, then the equation:

$$\dot{x} = Ax + h(t, x)$$

has $T$-periodic solution. This result is known, see [8] for its generalizations.

Example 2. The system of equations:

$$\begin{cases} \dot{x} = x^2 - y^2 + \delta\cos t \\ \dot{y} = -2xy + \delta\sin t \end{cases}$$

has $2\pi$-periodic solution, provided $\delta$ is small enough. It is a consequence of the fact that $\{0\}$ is the isolated invariant set of the index $-2$ for the dynamical system induced by the function $f(x, y) = (x^2 - y^2, -2xy)$ (see fig. 7, p. 18 in [4]).

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