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A Note on Tensor Products of Semi-Groups of Contractions

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There are several classical results concerning tensor products of unitary representations of groups. The most interesting case concerns LCA-groups. On the other hand, several semi-groups of contractions in Hilbert spaces have unitary dilations — see [5] for definitions and theorems. It appears then a natural question how the things are going with tensor products of such semi-groups, namely the question of their dilatability and the description of the properties of the corresponding unitary dilations. The present paper partly inspired by results of Berberian [2], contains some results in this field.

For the sake of completness we start with the summary of some known results concerning tensor products of semi-spectral measures and unitary representations (all spaces are complex). Then the convolution of operator measures in consideration appears. We follow here [2]. The second section of the present paper deals with the question of forming the spectral dilations of such convolutions. Next we describe unitary dilations of tensor products of positive definite operator functions in terms of dilations of the involved factors. In both cases the basic role is played by Naimark's dilations Theorems — [5], [7] for references. The two last essential parts of the paper deal with tensor products of semi-groups of contractions, which do have unitary dilations by Sz.-Nagy theorems as well some other ones unitarily dilatable semi-groups.

1. To begin with we recall shortly some definitions and some auxiliary theorems.

Suppose we are given a complex Hilbert space H. Vectors of H are denoted by $f, g, h, \ldots$ etc, the inner product in general by $(\cdot, \cdot)$ and the norm by $||\cdot||$ possibly by $(\cdot, \cdot)_H$, $||\cdot||_H$ in order to indicate H, simply because we shall deal with several different Hilbert spaces. $L(H)$ denotes the algebra of all linear bounded operators in H.

Let B be a σ-field of subsets of the space X. A function $F\colon B \to L(H)$ is called a semi-spectral measure if for every $f \in H$ the function $\mu_f(a) \overset{\text{df}}{=} (F(a)f, f)$ is a positive measure on B. It is evident that if $a_1, a_2 \in B$, then $a_1 \subset a_2$ implies $0 \leqslant F(a_1) \leqslant F(a_2) \leqslant F(X)$. The last inequality makes sense, because the values of F are selfadjoint positive operators. F is called normalized if $F(X) = I_H$ — the identity operator in H. If $F(a)$ is an orthogonal projection for each $a \in B$ and F is normalized then F is a spectral measure, namely $F(a_1 \cap a_2) = F(a_1)F(a_2)$ for $a_1, a_2 \in B$.

Recall — (see (1) for references) that if a complex, B-measurable function u is bounded almost everywhere with respect to F i.e. if $\sup\limits_{X-a}|u| < \infty$ for some $a \in B$, such that $F(a) = 0$, then the semi-spectral integral $\int\limits_X u dF$ makes sense, i.e. $\int\limits_X u d(Ff, g) = (\varphi(u)f, g)$ for some $\varphi(u) \in L(H)$ and all $f, g \in H$. Moreover $\|\varphi(u)\| \leqslant \sup_F \mathrm{ess}|u|$. If F is spectral, then $\varphi(u)$ is a spectral integral of Stone-von Neumann functional calculus.

Suppose X is a topological Hausdorff space. Let $K(X)$ be the totality of all compact subsets of X and $\mathcal{O}(X)$ the set of all open subsets of X. The semi-spectral measure F: $B(X) \to L(H)$ ($B(X)$ = the σ-field of Borel subsets of X) is called regular if for every $a \in B(X)$:

$$F(a) = \sup\{F(a_c)\colon\ a_c \in K(X), a_c \subset a\} = \inf\{F(a_0)\colon\ a_{\mathcal{O}} \in \mathcal{O}(X), a \subset a_{\mathcal{O}}\}\,.$$

Let F_i: $B(X_i) \to L(H_i)$ ($i = 1, 2$) be regular semi-spectral measures, where X_i are topological Hausdorff spaces. It is proved in (2) that there is the unique regular measure $F(\cdot) \in L(H_1 \otimes H_2)$ on the topological product $X_1 \times X_2$ defined on $B(X_1) \otimes B(X_2)$, the σ-field of subsets of $X_1 \times X_2$ generated by $a_1 \times a_2$, where $a_i \in B(X_i)$, such that for all $a_1 \in B(X_1)$, $a_2 \in B(X_2)$

$$F(a_1 \times a_2) = F(a_1) \otimes F(a_2)\,. \tag{1.0}$$

We call F the tensor product of F_1, F_2 and denote it simply by $F_1 \otimes F_2$. If F_1 and F_2 are spectral, so is $F_1 \otimes F_2$.

As it has been shown in [2], if

$$\mu^{F_i}_{f_i, g_i}(a_i) \overset{\mathrm{df}}{=} (F_i(a_i) f_i, g_i)_{H_i} \quad \text{for } i = 1, 2, \text{ then}$$

$$\mu^{F_1 \otimes F_2}_{f_1 \otimes f_2, g_1 \otimes g_2} = \mu^{F_1}_{f_1, g_1} \otimes \mu^{F_2}_{f_2, g_2}\,, \tag{1.1}$$

where the right hand side is just the suitable product scalar measure on $B(X_1 \times X_2)$.

Suppose that $X_1 = X_2 = X$ is an Abelian locally compact group, simply X is LCA. Define $\varphi(x, y) = x + y$.

DEFINITION 1.0 Let F_i be regular semi-spectral measures on the LCA group X. The *convolution* $F_1 * F_2$ is a semispectral measure defined by the formula

$$(F_1 * F_2)(a) = (F_1 \otimes F_2)(\varphi^{-1}(a))\,, \quad \text{for } a \in B(X)\,.$$

It is not difficult to prove, see (2), that

(1.2) The convolution $F_1 * F_2$ of Def. 1.0 is a regular measure on $B(X)$; if F_1, F_2 are spectral measures, so is $F_1 * F_2$; and

$$\mu^{F_1 * F_2}_{f_1 \otimes f_2, g_1 \otimes g_2} = \mu^{F_1}_{f_1, g_1} * \mu^{F_2}_{f_2, g_2}\,, \tag{1.3}$$

where the measure on the right-hand side of the above formula is an ordinary convolution of scalar measures appearing in (1.1) — we refer for the Fourier analysis on LCA groups to Rudin [8].

Let $\hat{X}$ be the dual group of X and $\langle s, x\rangle$ the values of $s \in \hat{X}$ at $x \in X$. The Fourier transform of the regular measure μ on X is defined as

$$\hat{\mu}(x) = \int_{\hat{X}} \langle s, x\rangle d\mu_s$$

Next

$$\widehat{\mu_1 * \mu_2} = \hat{\mu}_1 \cdot \hat{\mu}_2$$

for regular measures μ_1, μ_2 on X.

It follows now from (1.1) and (1.3) that

$$\int_{\hat{X}} \langle s, x\rangle d_s(F_1 * F_2) = \int_{\hat{X}} \langle s, x\rangle dF_1(s) \otimes \int_{\hat{X}} \langle s, x\rangle dF_2(s) \tag{1.4}$$

for $x \in X$. If F_1 and F_2 are spectral measures corresponding to unitary representations $U_i(\cdot)$: $X \to L(H_i)$ $(i = 1, 2)$, then

$$U_1(x) \otimes U_2(x) = \int_{\hat{X}} \langle s, x\rangle ds(F_1 * F_2), \tag{1.5}$$

see [2].

2. In all what follows we refer for terminology, definitions and proofs to Mlak [5] as well as to the original papers quoted in [7].

We start with Naimark's theorem see [5] for references.

(N_1) *Let F: $B(X) \to L(H)$ be a semi-spectral measure defined on a σ-field $B(X)$ of subsets of X. Then there is a Hilbert space K, a linear bounded operator V: $H \to K$ and a spectral measure $E(\cdot)$: $B(X) \to L(K)$ such that*

2.0) *$F(a) = V^* E(a) V$ for $a \in B(X)$.*
Moreover, if K is the minimal dilation space i.e. if $K = \bigvee_{a \in B(X)} E(a) VH$, then the triple (E, K, V) is determined up to unitary equivalence and (2,0) *is called the canonical representation of F and E the minimal spectral V dilation of F. If F is normalized then* (2.0) *takes the form*

(2.1) *$F(a)f = PE(a)f$, $a \in B(X)$, $f \in H$,*
where $H \subset K$ and P is orthogonal projection of K onto H and the minimality of K means that $K = \bigvee_{a \in B(X)} E(a) H$. We then say that F is a projection at E i.e. the minimal spectral dilation of F.

COROLLARY 2.0 *If X is a Hausdorff topological space and F is regular then E is regular.*

Let X_i $(i = 1, 2)$ be Hausdorff topological spaces; we shall now prove the following (theorem:

THEOREM 2.0 *Let F_i: $B(X_i) \to L(H_i)$, $(i = 1, 2)$ be regular semi-spectral measures and let $F_i(a_i) = V_i^* E_i(a_i) V_i$ $(i = 1, 2)$, $a_i \in B(X_i)$ be their canonical representations. Then the canonical representation of $F_1 \otimes F_2$ is*

(2.2) $F_1 \otimes F_2 = (V_1 \otimes V_2)^*(E_1 \otimes E_2)(V_1 \otimes V_2)$
and consequently, the corresponding minimal dilation space $K = K_1 \otimes K_2$, *where* K_i *is the minimal dilation space for* F_i $(i = 1, 2)$.

Proof:
Let $a_i \in B(X_i)$; $f_i, g_i \in H_i$ for $i = 1, 2$. Then

$$\begin{aligned}
&((F_1 \otimes F_2)(a_1 \times a_2) f_1 \otimes f_2, g_1 \otimes g_2) = (F_1(a_1) f_1, g_1)(F_2(a_2) f_2, g_2) \\
&= (E_1(a_1) V_1 f_1, V_1 g_1)(E_2(a_2) V_2 f_2, V_2 g_2) = \mu^{E_1}_{V_1 f_1, V_1 g_1}(a_1) \mu^{E_2}_{V_2 f_2, V_2 g_2}(a_2) \\
&= \mu^{E_1 \otimes E_2}_{V_1 f_1 \otimes V_2 f_2, V_1 g_1 \otimes V_2 g_2}(a_1 \times a_2) \\
&= ((E_1 \otimes E_2)(a_1 \times a_2)(V_1 \otimes V_2)(f_1 \otimes f_2), (V_i \otimes V_2)(g_1 \otimes g_2)) \\
&= ((V_1 \otimes V_2)^*(E_1 \otimes E_2)(a_1 \times a_2)(V_1 \otimes V_2)(f_1 \otimes f_2), g_1 \otimes g_2))
\end{aligned}$$

by (1.0) and (1.1). Since elements $f \otimes g$, $f \in H_1$ $g \in H_2$ span $H_1 \otimes H_2$, the formula (2.2) follows by regularity of F_1, F_2 and from the statements related to (1.0).

In order to proof the minimality property we must show that

$$(2\,3) \qquad \bigvee_{a \in B(X_1) \otimes B(X_2)} (E_1 \otimes E_2)(a)(V_1 \otimes V_2)(H_1 \otimes H_2) = K_1 \otimes K_2$$

Plainly:

$$(2.4) \qquad \bigvee_{a_i \in B(X_i)} (E_1 \otimes E_2)(a_1 \times a_2)(V_1 \otimes V_2)(H_1 \otimes H_2) \subset K_1 \otimes K_2, \qquad i = 1, 2$$

where $K_1 \otimes K_2$ is the closed linear span of elements $E_1(a_1) V_1 f_1 \otimes E_2(a_2) V_2 f_2$. Since $(E_1 \otimes E_2)(a) \in L(K_1 \otimes K_2)$, the left-hand side of 2.3 is included in $K_1 \otimes K_2$ and includes the left-hand side of 2.4, which completes the proof.

The second celebrated Naimark's theorem, see [5] for references, reads as follows:

(N$_2$) *Let* $T(\cdot)$: $G \to L(H)$ *be a positive definite function on the group* G. *Then there is a Hilbert space* K *and a linear bounded operator* V: $H \to K$ *such that*

$$(2.5) \qquad T(g) = V^* U(g) V \quad (\text{for all } g \in G),$$

where $U(\cdot)$: $G \to L(K)$ *is a unitary representation of* G, *called a* V*-unitary dilatation of* $T(\cdot)$. *If* K *is minimal i.e.* $K = \bigvee_{g \in G} U(g) VH$ *then the triple* $(K, V, U(\cdot))$ *is determined up to unitary equivalence. If* $T(\cdot)$ *is unital i.e.* $T(e) = I_H$ (e *is a unit of* G), *then* (2.5) *reads as follows*:

$$(2.6) \qquad T(g) f = PU(g) f, \quad g \in G, \quad f \in H,$$

where $H \subset K$, P *is orthogonal projection of* K *onto* H *and then the minimal* $K = \bigvee_{g \in G} U(g) H$.

The formula (2.5) *is called the canonical representation of* T *if* K *is minimal.*

Remark 2.0 *If* G *is commutative, then the positive definitness of* $T(\cdot)$ *is equivalent to the positive definitness of the function* $\varphi_f(g) = (T(g) f, f)$ *for every* f — *see* [5].

Remark 2.1. *If G is a topological group and $T(\cdot)$ is weakly continuous at $g = e$, then $T(\cdot)$ and $U(\cdot)$ are strongly continuous* — see [5].

Using arguments even simpler then in the proof of theorem 2.0 one proves easily the following

THEOREM 2.1. *Let $T_i(g) \in L(H_i)$ $(i = 1, 2)$ be two positive definite functions on the group G with canonical representations: $T_i(g) = V_i^* U_i(g) V_i$ $(i = 1, 2)$ and minimal spaces K_1, K_2. Then the function $T(g) = T_1(g) \otimes T_2(g)$ is positive definite and has the cannonical representation*

$$T(g) = (V_1 \otimes V_2)^* (U_1(g) \otimes U_2(g))(V_1 \otimes V_2) \quad \textit{for } g \in G \tag{2.7}$$

with the minimal space $K = K_1 \otimes K_2$.

3. Let G be a group and S a unital ($e \in S$) subsemigroup of G. Suppose $T(s) \in L(H)$ is defined for $g \in S$.

DEFINITION 3.0. We say that $T(s)$, $s \in S$ is *S-unitary dilatable* if there is a Hilbert space K, a linear bounded operator V: $H \to K$ and a unitary representation $U(\cdot)$: $G \to$ $\to L(K)$ such that

$$T(s) = V^* U(s) V \quad \text{for } s \in S \tag{3.0}$$

Notice that if $T(e) = I_H$ then (3.0) reduces to formula

$$T(s)_k = PU(s)f \quad s \in S, f \in H, \tag{3.1}$$

where P is the orthogonal projection of K onto $H \subset K$. Then we say that $T(\cdot)$ is an S-projection of $U(\cdot)$; we then write $T(s) = \operatorname{pr} U(s)$, $s \in S$. Notice that if S_i is a subsemigroup of G_i, e_i is a unit of G_i $(i = 1, 2)$ then $S_1 \times S_2 \subset G_1 \times G_2$ is a unital subsemigroup of $G_1 \times G_2$ with coordinatewise defined operations.

Remark 3.0 *The function $T(\cdot)$ is S-unitarily dilatable if and only if it has a positive definite extension on the whole group G.*

The following theorem is now in order:

THEOREM 3.0 *Let the operators $T_i(s_i) \in L(H_i)$ $(i = 1, 2)$ be defined on unital subsemigroups S_1, S_2 of groups G_1, G_2 respectively. If $T_i(s_i)$ is S_i-unitarily dilatable, say $T_i(s_i) = V_i^* U_i(s_i) V_i$ for $s_i \in S_i$, $U_i(\cdot)$ being suitable unitary representations, then $T(s_1, s_2)$ $\overset{\text{df}}{=} T_1(s_1) \otimes T_2(s_2)$ is $S_1 \times S_2$-unitary dilatable and*

$$T_1(s_1) \otimes T_2(s_2) = (V_1 \otimes V_2)^* U_1(s_1) \otimes U_2(s_2)(V_1 \otimes V_2) \tag{3.2}$$

$$\textit{for } s_i \in S_i \ (i = 1, 2)$$

Proof:
We have for $s_i \in S_i$, $f_i \in H_i$:

$$\begin{aligned}(T_1(s_1)\otimes T_2(s_2))(f_1\otimes f_2) &= (V_1^* U_1(s_1) V_1 f_1)\otimes(V_2^* U_2(s_2) V_2 f_2)\\ &= ((V_1^* U_1(s_1) V_1)\otimes(V_2^* U_2(s_2) V_2))(f_1\otimes f_2)\\ &= (V_1^*\otimes V_2^*)(U_1(s_1) V_1 f_1\otimes U_2(s_2) V_2 f_2)\\ &= (V_1\otimes V_2)^*(U_1(s_1)\otimes U_2(s_2))(V_1\otimes V_2)(f_1\otimes f_2)\end{aligned}$$

as was to be proved, because in general $(V_1\otimes V_2)^* = V_1^*\otimes V_2^*$.

It follows now from Naimark's theorems and for G of class LCA and formula (1.4), and theorems (2.0), (2.1), (3.0) that for $T_i(\cdot)$: $S \to L(H_i)$ $(i = 1, 2, \dots, p)$, where S is a unital subsemigroup of G which are S-projections of unitary representations $U_i(\cdot)$ of G we get, by an easy induction, the main result, namely the basic formula:

(B) $$\begin{aligned} T_1(x)\otimes T_2(x)\otimes \dots \otimes T_p(x) &= \mathrm{pr}(U_1(x)\otimes U_2(x)\otimes \dots \otimes U_p(x))\\ &= \int_{\hat{G}} \langle s, x\rangle d_s(F_1 * \dots * F_p) \quad \text{for } x\in S\end{aligned}$$

where $T_i(x) = \int_{\hat{G}} \langle s, x\rangle dF_i(s)$, $x\in S$ $(i = 1, 2, \dots, p)$

and the spectral dilation of F_i is the spectral measure of $U_i(\cdot)$. Formula B is a far going generalization of formulae appearing in theorems of Berberian's paper, namely theorem 2 in case $S = Z_+$ = additive semigroup of nonnegative integers, $T(x) = T^X$, $\|T\| \leqslant 1$. A more general version of (B) appears with suitable V-s operators just like in theorem 2.1, when combined with Stone's theorem for unitary representations of LCA groups.

Let us recall now some unitarily S-dilatable semigroups of contractions, in fact the ones that are S-projections.

(3.3) Let G be a discrete subgroup of the additive group $\boldsymbol{R}$ of reals. Suppose $S = G\cap \boldsymbol{R}$ and let $T(\cdot)$ be a semigroup of contractions over S i.e. $T(0) = I_H$, $T(t+s) = T(t)T(s)$ for $t, s\in S$ and $\|T(t)\| \leqslant 1$. Then $T(\cdot)$ is S-unitarily dilatable, namely has S-projection i.e. $T(s) = \mathrm{pr}\, U(s)$ for $s\in S$, where $U(\cdot)$ is a unitary representation of G.

The above result of [4] generalizes the classical results of Sz.-Nagy see [7], namely when

(3.4) $G = \boldsymbol{Z}$ — the additive group of integers; then $S = \boldsymbol{Z}_+$ = the semigroup of nonnegative integers and $T(s) = T^s$ when $s\in \boldsymbol{Z}_+$, T being a fixed contraction

and

(3.5) $G = \boldsymbol{R}$ with the usual topology and $S = [0, +\infty)$ and $T(s)$ $(s \geqslant 0)$ is a strongly continuous one parameter semigroup of contractions.

In case (3.3) we have

(A$_1$) $$T(x) = \int_{\hat{G}} \langle s, x\rangle dF_s \text{ for } x\in S$$

with normalized semispectral measure F on $\hat{G}$.