

On the Theorem of Fatou for the Biharmonic Problem in a Half-space

by Stanisław BURYS and Elżbieta POLIT-GANCARCZYK

In the paper [1] the solution of the classical biharmonic problem for a half space has been obtained. In our paper we shall generalize this result to the case of boundary conditions given by signed measures. A similar problem for the n -dimensional ball has been solved by E. Keller in the paper [3]. Before the formulation of the problem we shall give necessary definitions and notations. Points of the real n -dimensional Euclidean space \mathbf{R}^n ($n \geq 2$) are denoted by $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n)$, the inner product and the norm in \mathbf{R}^n are defined by $(x|y) := \sum_{i=1}^n x_i y_i$ and $\|x\| := \sqrt{(x|x)}$, respectively, where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$. We shall use notations:

$$\mathbf{R}_0^n := \{(x_1, \dots, x_{n-1}, 0) \in \mathbf{R}^n\} \quad \text{and} \quad \mathbf{R}_+^n := \{(x', x_n) \in \mathbf{R}^n, x_n > 0\}.$$

$L(\mathbf{R}^{n-1})$ denotes the class of Borel subsets of \mathbf{R}^{n-1} .

Let $\xi = (\xi', 0) \in \mathbf{R}_0^n$, then

$$\Gamma(\xi, a) := \left\{ x = (x', x_n) \in \mathbf{R}_+^n : \|(x', 0) - \xi\| < a \cdot x_n, 0 < a = \operatorname{tg} \alpha, \alpha < \frac{\pi}{2} \right\}$$

and R_ξ denotes the collection of balls $K \subset \mathbf{R}^{n-1}$ containing the point $\xi' \in \mathbf{R}^{n-1}$. Let μ be a signed measure, defined on $L(\mathbf{R}^{n-1})$, which is finite on compact sets and $\mu(\emptyset) = 0$.

The limit

$$\lim_{\substack{K \in R_\xi \\ \mathcal{O}_{n-1}(K) \rightarrow 0}} \frac{\mu(K)}{\mathcal{O}_{n-1}(K)},$$

if it exists and is independent of the choice of the balls K , is called *the derivative of measure μ at the point ξ'* and is denoted by $(D\mu)(\xi')$, where $\mathcal{O}_{n-1}(K)$ denotes the $(n-1)$ -dimensional Lebesgue measure of set K .

We observe that

$$(D\mu)(\xi') = \frac{1}{\tau_{n-1}} \lim_{\substack{r \rightarrow 0 \\ K \in R_\xi}} \frac{\mu(K)}{r^{n-1}},$$

where τ_{n-1} denotes the volume of the $(n-1)$ -dimensional unit ball and r denotes the radius of the ball K .

If a point ξ' is the center of balls $K \in R_\xi$, then $(D\mu)(\xi')$ is called *the symmetric derivative* and is denoted by $(D_s\mu)(\xi')$. It is obvious that if $(D\mu)(\xi')$ exists, $(D_s\mu)(\xi')$ exists too, and $(D\mu)(\xi') = (D_s\mu)(\xi')$.

The limit

$$\lim_{\substack{r \rightarrow 0 \\ K \in R_\xi K \\ \xi \in R_0^n}} \frac{\mu(K)}{\int_{\| \xi - (y', 0) \|} d\theta_{n-1}(y')},$$

if it exists, is called *the second symmetric derivative of measure μ at a point ξ'* , where ξ' is the center of the balls K . It is denoted by $(D_s^2\mu)(\xi')$.

We have (see [4] page 36)

$$(D_s^2\mu)(\xi') = \frac{n}{\theta_{n-1}} \lim_{r \rightarrow 0} \frac{\mu(K)}{r^n},$$

where θ_n denotes the surface area of the unit sphere in \mathbf{R}^n .

The radial limit $R\text{-}\lim_{x \rightarrow \xi} u(x)$ of a function u defined on \mathbf{R}^n at the point $\xi = (\xi', 0) \in \mathbf{R}_0^n$ is defined as follows:

$$R\text{-}\lim_{x \rightarrow \xi} u(x) := \lim_{x_n \rightarrow 0^+} u(\xi', x_n).$$

If $\lim_{\substack{x \rightarrow \xi \\ x \in \Gamma(\xi, a)}} u(x)$ exists for every $a > 0$ and does not depend on a , then we denote it by $\theta\text{-}\lim_{x \rightarrow \xi} u(x)$ and we call it a *nontangential limit*.

Let signed measure μ and η be given, defined on $L(\mathbf{R}^{n-1})$.

ASSUMPTION A. μ and η are of bounded variation.

ASSUMPTION B. There are non-negative constants C_1, C_2 and constants $\beta_1, \beta_2 \in (n-1, n)$ such that $|\mu(K)| \leq C_1 r^{\beta_1}$ $|\eta(K)| \leq C_2 r^{\beta_2}$, for every ball $K \subset \mathbf{R}^{n-1}$ with sufficiently large radius r .

Let

$$P(\mu, \eta, \mathbf{R}^{n-1})(x) := \frac{x_n^2}{\alpha_n} \int_{\mathbf{R}^{n-1}} \frac{1}{\|x - (y', 0)\|^n} d\mu(y') + \frac{n}{\alpha_n} x_n^3 \int_{\mathbf{R}^{n-1}} \frac{1}{\|x - (y', 0)\|^{n+2}} d\eta(y')$$

for $x = (x', x_n) \in \mathbf{R}_+^n$, where

$$\alpha_n := \int_{\mathbf{R}^{n-1}} \frac{dt_1 \dots dt_{n-1}}{(t_1^2 + t_2^2 + \dots + t_{n-1}^2 + 1)^{n/2}}.$$

We shall prove that under Assumptions A and B and under suitable assumptions with respect to measures μ and η we have:

a) $\Delta^2 P(\mu, \eta, \mathbf{R}^{n-1})(x) = 0$ for $x \in \mathbf{R}_+^n$,

$$\text{b) } \theta\text{-}\lim_{x \rightarrow \xi = (\xi', 0)} P(\mu, \eta, \mathbf{R}^{n-1})(x) = (D\eta)(\xi'),$$

$$\text{c) } \theta\text{-}\lim_{x \rightarrow \xi = (\xi', 0)} \frac{\partial}{\partial x_n} P(\mu, \eta, \mathbf{R}^{n-1})(x) = (D\mu)(\xi') \text{ for a.e. } \xi \in \mathbf{R}_0^n.$$

It is easy to check that under Assumptions A and B functions

$$\frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \frac{x_n^2}{\|x - (y', 0)\|^n},$$

$$\frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \frac{x_n^3}{\|x - (y', 0)\|^{n+2}},$$

where $k = 0, 1, 2, 3, 4$, $k = k_1 + \dots + k_n$ and k_1, \dots, k_n non-negative integer, satisfy the assumptions of Theorem 132, Chapter IV, of [8]. Thus

$$\begin{aligned} & \frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \left(x_n^2 \int_{\mathbf{R}^{n-1}} \frac{1}{\|x - (y', 0)\|^n} d\mu(y') \right) \\ &= \int_{\mathbf{R}^{n-1}} \frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \frac{x_n^2}{\|x - (y', 0)\|^n} d\mu(y'), \\ (1)' \quad & \frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} x_n^3 \int_{\mathbf{R}^{n-1}} \frac{1}{\|x - (y', 0)\|^{n+2}} d\eta(y') \\ &= \int_{\mathbf{R}^{n-1}} \frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \frac{x_n^3}{\|x - (y', 0)\|^{n+2}} d\eta(y'). \end{aligned}$$

Hence and from the fact that functions $\frac{x_n^2}{\|x - (y', 0)\|^n}$ and $\frac{x_n^3}{\|x - (y', 0)\|^{n+2}}$, for fixed $y' \in \mathbf{R}^{n-1}$, are biharmonic in \mathbf{R}_+^n as functions of variable x , it follows that

$$\Delta^2 P(\mu, \eta, \mathbf{R}^{n-1})(x) = 0$$

for $x \in \mathbf{R}_+^n$. Then the condition a) is proved.

To prove conditions b) and c) we first show some lemmas. Let us denote

$$P_1(\mu, \mathbf{R}^{n-1})(x) := \frac{x_n^2}{\alpha_n} \int_{\mathbf{R}^{n-1}} \frac{1}{\|x - y\|^n} d\mu(y')$$

$$P_2(\eta, \mathbf{R}^{n-1})(x) := \frac{n}{\alpha_n} x_n^3 \int_{\mathbf{R}^{n-1}} \frac{1}{\|x - y\|^{n+2}} d\eta(y')$$

$$P_3(\mu, \mathbf{R}^{n-1})(x) := \frac{2}{\alpha_n} x_n \int_{\mathbf{R}^{n-1}} \frac{1}{\|x - y\|^n} d\mu(y')$$

$$P_4(\mu, \mathbf{R}^{n-1})(x) := \frac{n}{\alpha_n} x_n^3 \int_{\mathbf{R}^{n-1}} \frac{1}{\|x-y\|^{n+2}} d\mu(y')$$

$$P_5(\eta, \mathbf{R}^{n-1})(x) := \frac{3n}{\alpha_n} x_n^2 \int_{\mathbf{R}^{n-1}} \frac{1}{\|x-y\|^{n+2}} d\eta(y')$$

$$P_6(\eta, \mathbf{R}^{n-1})(x) := \frac{n(n+2)}{\alpha_n} x_n^4 \int_{\mathbf{R}^{n-1}} \frac{1}{\|x-y\|^{n+4}} d\eta(y'),$$

where $y = (y', 0) \in \mathbf{R}_0^n$.

Hence

$$P(\mu, \eta, \mathbf{R}^{n-1})(x) = P_1(\mu, \mathbf{R}^{n-1})(x) + P_2(\eta, \mathbf{R}^{n-1})(x)$$

and by (1)'

$$\begin{aligned} \frac{\partial}{\partial x_n} P(\mu, \eta, \mathbf{R}^{n-1})(x) &= P_3(\mu, \mathbf{R}^{n-1})(x) - P_4(\mu, \mathbf{R}^{n-1})(x) \\ &\quad + P_5(\eta, \mathbf{R}^{n-1})(x) - P_6(\eta, \mathbf{R}^{n-1})(x). \end{aligned}$$

LEMMA 1. Let

$$I(x_n) := x_n^\alpha \int_0^\infty \frac{r^{n+\gamma}}{(x_n^2 + r^2)^{\frac{n+\beta}{2}}} dr,$$

$$\text{where } \begin{cases} \alpha = 1, 2, 3, 4, \\ \beta = 2, 4, 6, \text{ then} \\ \gamma = 0, 1, \\ \beta - \gamma > 1, \end{cases}$$

$$(a) \lim_{x_n \rightarrow 0^+} I(x_n) = 0$$

for $\alpha - \beta + \gamma + 1 > 0$,

$$b) I(x_n) = \text{constant}$$

for $\alpha - \beta + \gamma + 1 = 0$.

Proof. Substituting $r = x_n \cdot z$, we have

$$I(x_n) = x_n^{\alpha - \beta + \gamma + 1} \int_0^\infty \frac{z^{n+\gamma}}{(1+z^2)^{n+\beta/2}} dz,$$

but

$$\int_0^\infty \frac{z^{n+\gamma}}{(1+z^2)^{n+\beta/2}} dz < \infty,$$

80

- a) $\lim_{x_n \rightarrow 0^+} I(x_n) = 0$ for $\alpha - \beta + \gamma + 1 > 0$,
 (b) $I(x_n)$ constant for $\alpha - \beta + \gamma + 1 = 0$.

LEMMA 2. Let μ and η be signed measures on $L(\mathbf{R}^{n-1})$. We assume Assumption A and B for μ and η and we assume that

$$(D_s \mu)(\xi') = (D_s \eta)(\xi') = 0.$$

Then

$$R\text{-}\lim_{x \rightarrow \xi} P(\mu, \eta, \mathbf{R}^{n-1})(x) = 0,$$

for $x \in \mathbf{R}_+^n$.

If we assume additionally that $(D_s^2 \eta)(\xi') = 0$, then

$$R\text{-}\lim_{x \rightarrow \xi} \frac{\partial}{\partial x_n} P(\mu, \eta, \mathbf{R}^{n-1})(x) = 0.$$

Proof. We shall prove the second part of proposition on an example of integrals P_3 and P_5 . The proof of the first part follows analogously.

Let ε be an arbitrary positive number. From the assumption $(D_s \mu)(\xi') = 0$ it follows that there exists a positive number $r_0 = r_0(\varepsilon)$ such that

$$(1) \quad \left| \frac{\mu(K)}{\tau_{n-1} r^{n-1}} \right| < \varepsilon$$

for $r < r_0$, where $K = K_{\xi', r}$. Hence we have

$$(2) \quad |\mu(K)| < \varepsilon \tau_{n-1} r^{n-1}$$

for $0 < r < r_0$.

Let

$$(3) \quad P_3(\mu, \mathbf{R}^{n-1})(x) = P_3(\mu, K_{\xi', r_0})(x) + P_3(\mu, \mathbf{R}^{n-1} \setminus K_{\xi', r_0})(x)$$

and let

$$F(r) := \int_{K_{\xi', r}} d\mu(y') = \mu(K_{\xi', r}).$$

Then we have

$$P_3(\mu, K_{\xi', r_0})(\xi', x_n) = \frac{2}{\alpha_n} x_n \int_{K_{\xi', r_0}} \frac{1}{\|(\xi', x_n) - y'\|^n} d\mu(y') = \frac{2}{\alpha_n} x_n \int_0^{r_0} \frac{1}{(x_n^2 + r^2)^{n/2}} dF(r).$$

Applying the integration by parts we get

$$P_3(\mu, K_{\xi', r_0})(\xi', x_n) = x_n \frac{F(r_0)}{(x_n^2 + r_0^2)^{n/2}} + n \cdot x_n \int_0^{r_0} \frac{r F(r)}{(x_n^2 + r^2)^{n+2/2}} dr.$$

From (2) it follows that

$$|F(r)| < \tau_{n-1} r^{n-1} \varepsilon$$

for $0 < r < r_0$, so

$$\begin{aligned} \left| nx_n \int_0^{r_0} \frac{rF(r)}{(x_n^2 + r^2)^{n+2/2}} dr \right| &\leq nx_n \int_0^{r_0} \frac{rF(r)}{(x_n^2 + r^2)^{n+2/2}} dr \\ &\leq n\tau_{n-1}\varepsilon x_n \int_0^{r_0} \frac{r^n}{(x_n^2 + r^2)^{n+2/2}} dr \leq n\tau_{n-1}\varepsilon x_n \int_0^{\infty} \frac{r^n}{(x_n^2 + r^2)^{n+2/2}} dr, \end{aligned}$$

but in view of Lemma 1

$$x_n \int_0^{\infty} \frac{r^n}{(x_n^2 + r^2)^{n+2/2}} dr = \text{const} = C,$$

hence

$$(4) \quad |P_3(\mu, K_{\xi', r_0})(\xi', x_n)| \leq x_n \frac{|F(r_0)|}{(x_n^2 + r_0^2)^{n/2}} + \varepsilon \tau_{n-1} n C.$$

On the other hand, by Assumption B we get

$$\begin{aligned} \left| \int_{\mathbf{R}^{n-1} \setminus K_{\xi', r_0}} \frac{1}{\|(\xi', x_n) - (y', 0)\|^n} d\mu(y') \right| &\leq \int_{\mathbf{R}^{n-1} \setminus K_{\xi', r_0}} \frac{1}{\|(\xi', 0) - (y', 0)\|^n} d|\mu|(y') \\ &\leq \int_{r_0}^{\infty} \frac{1}{r^n} dF^*(r) < C_3, \end{aligned}$$

for an arbitrary fixed $x = (\xi', x_n) \in \mathbf{R}_+^n$, where $F^*(r) = |\mu|(K_{\xi', r})$ and C_3 is a positive constant. Hence

$$(5) \quad |P_3(\mu, \mathbf{R}^{n-1} \setminus K_{\xi', r_0})(\xi', x_n)| \leq x_n C_3.$$

By (3), (4) and (5) we get

$$|P_3(\mu, \mathbf{R}^{n-1})(\xi', x_n)| \leq \varepsilon \tilde{C}_3$$

for $0 < x_n < \delta$, where \tilde{C}_3 is a positive constant and δ is a sufficiently small positive number.

Hence

$$R\text{-}\lim_{x \rightarrow \xi} P_3(\mu, \mathbf{R}^{n-1})(x) = 0.$$

We shall prove now that

$$R\text{-}\lim_{x \rightarrow \xi} P_5(\eta, \mathbf{R}^{n-1})(x) = 0.$$

By the assumption $(D_s^2 \eta)(\xi') = 0$, so

$$\left| \frac{n\eta(K)}{\theta_{n-1}r^n} \right| < \varepsilon$$

i.e.

$$|\eta(K)| < \frac{\varepsilon}{n} \theta_{n-1} r^n$$

for $0 < r < r_1$, where r_1 is some positive number. Denoting

$$G(r) := \int_{K_{\xi', r}} d\eta(y') = \eta(K_{\xi', r})$$

we get

$$|G(r)| < \frac{\varepsilon}{n} \theta_{n-1} r^n.$$

We have

$$P_5(\eta, K_{\xi', r_1})(\xi', x_n) = x_n^2 \frac{3n}{\alpha_n} \int_{K_{\xi', r_1}} \frac{1}{\|x - (y', 0)\|^{n+2}} d\eta(y') = x_n^2 \int_0^{r_1} \frac{1}{(x_n^2 + r^2)^{n+2/2}} dG(r).$$

Applying integration by parts we get

$$P_5(\eta, K_{\xi', r_1})(\xi', x_n) = x_n^2 \frac{G(r_1)}{(x_n^2 + r_1^2)^{n+2/2}} + (n+2)x_n^2 \int_0^{r_1} \frac{rG(r)}{(x_n^2 + r^2)^{n+4/2}} dr.$$

It is

$$\begin{aligned} \left| (n+2)x_n^2 \int_0^{r_1} \frac{rG(r)}{(x_n^2 + r^2)^{n+4/2}} dr \right| &\leq (n+2)x_n^2 \int_0^{r_1} \frac{r|G(r)|}{(x_n^2 + r^2)^{n+4/2}} dr \\ &\leq \frac{n+2}{n} \theta_{n-1} \varepsilon x_n^2 \int_0^{\infty} \frac{r^{n+1}}{(x_n^2 + r^2)^{n+4/2}} dr, \end{aligned}$$

but in view of Lemma 1

$$\frac{n+2}{n} x_n^2 \theta_{n-1} \int_0^{\infty} \frac{r^{n+1}}{(x_n^2 + r^2)^{n+4/2}} dr = \text{constant} = C_4,$$

so

$$|P_5(\eta, K_{\xi', r_1})(\xi', x_n)| \leq x_n^2 \frac{|G(r_1)|}{(x_n^2 + r_1^2)^{n+2/2}} + C_4 \varepsilon.$$

Further we prove similarly as for integral P_3 .

LEMMA 3. We assume that signed measures μ, η defined on $L(\mathbf{R}^{n-1})$ satisfy Assumptions **A** and **B** and they are still nonpositive or non-negative. Moreover we assume that $(D_s \mu)(\xi') = (D_s \eta)(\xi') = 0$.

Then

$$1^\circ \quad \theta - \lim_{x \rightarrow \xi} P(\mu, \eta, \mathbf{R}^{n-1})(x) = 0$$

If we assume additionally that $(D_s^2|\eta|)(\xi') = 0$, then

$$2^\circ \quad \theta - \lim_{x \rightarrow \xi} \frac{\partial}{\partial \lambda_\eta} P(\mu, \eta, \mathbf{R}^{n-1})(x) = 0.$$

Proof. We shall carry out the proof for integral $P_1(\mu, \mathbf{R}^{n-1})$, for other integrals the proof being analogous.

Let measures μ and η non-negative. We are to prove that

$$\theta - \lim_{x \rightarrow \xi} P_1(\mu, \mathbf{R}^{n-1})(x) = \theta - \lim_{x \rightarrow \xi} \left(x_n^2 \int_{\mathbf{R}^{n-1}} \frac{1}{\|x - (y', 0)\|^n} d\mu(y') \right) = 0.$$

Let us denote the axis of the cone $\Gamma(\xi, a)$ by p and by w we denote the projection of the point x on the line p . We have

$$\frac{\|w - x\|}{\|w - \xi\|} \leq a$$

for $x \in \Gamma(\xi, a)$, so

$$\|w - y\| \leq \|w - x\| + \|x - y\| \leq a\|w - \xi\| + \|x - y\| \leq (a+1)\|x - y\|$$

and therefore there exists a constant $C_5 > 0$ such that

$$\frac{1}{\|x - y\|} \leq C_5 \frac{1}{\|w - y\|}$$

for every $x \in \Gamma(\xi, a)$ and $y \in \mathbf{R}^{n-1}$. We have now

$$\int_{\mathbf{R}^{n-1}} \frac{1}{\|x - y\|^n} d\mu(y') \leq C_6 \int_{\mathbf{R}^{n-1}} \frac{1}{\|w - y\|^n} d\mu(y'),$$

where $C_6 := C_5^n$, hence

$$\begin{aligned} 0 \leq \theta - \lim_{x \rightarrow \xi} \left(x_n^2 \int_{\mathbf{R}^{n-1}} \frac{1}{\|x - y\|^n} d\mu(y') \right) &\leq C_6 \theta - \lim_{x \rightarrow \xi} \left(x_n^2 \int_{\mathbf{R}^{n-1}} \frac{1}{\|w - y\|^n} d\mu(y') \right) \\ &= C_6 R - \lim_{w \rightarrow \xi} \left(x_n^2 \int_{\mathbf{R}^{n-1}} \frac{1}{\|w - y\|^n} d\mu(y') \right) = 0, \end{aligned}$$

but in view of Lemma 2 it is

$$\theta - \lim_{x \rightarrow \xi} \left(x_n^2 \int_{\mathbf{R}^{n-1}} \frac{1}{\|x - y\|^n} d\mu(y') \right) = 0.$$

For non-positive measures μ and η the proof is similar.

We say that *measure* η (defined on $L(\mathbf{R}^{n-1})$) fulfils condition D at a point $\xi' \in \mathbf{R}^{n-1}$ if and only if there exists a function f defined on \mathbf{R}^{n-1} such that

1° f is differentiable on \mathbf{R}^{n-1} and its first derivatives are continuous at the point $\xi' \in \mathbf{R}^{n-1}$,

2° $f(y') = O(r^\alpha)$ for $r \rightarrow \infty$, where $r = \|y'\|$, $\alpha \in (0, 1)$,

3° the measure

$$\lambda(B) := \eta(B) - \int_B f(y') dy'$$

($B \in L(\mathbf{R}^{n-1})$, $(\eta(B) < \infty)$), satisfies the condition $(D_s^2|\lambda|)(\xi') = 0$.

LEMMA 4. We assume that

1° *measure* η satisfies condition D at the point $\xi' \in \mathbf{R}^{n-1}$,

2° there exists the bounded symmetric derivative $(D_s\mu)(\xi') = b$,

3° *measures* μ and η satisfy Assumption A and B ,

4° $(D_s|\delta|)(\xi') = 0$,

where $\delta(B) := \mu(B) - b \int_B dy'$, $B \in L(\mathbf{R}^{n-1})$, $\mathcal{O}_{n-1}(B) < \infty$. Then

$$\theta - \lim_{x \rightarrow \xi} \frac{\partial}{\partial x_n} P(\mu, \eta, \mathbf{R}^{n-1})(x) = b.$$

Proof. Let f be the function given by the condition D . Let us denote

$$I := \theta - \lim_{x \rightarrow \xi} \frac{\partial}{\partial x_n} P(\mu, \eta, \mathbf{R}^{n-1})(x) - b.$$

We need to proof that $I = 0$.

Similarly as in [2] it can be proved that

$$b = \lim_{x \rightarrow \xi} \frac{\partial}{\partial x_n} P(f, b, \mathbf{R}^{n-1})(x),$$

where

$$P(f, b, \mathbf{R}^{n-1})(x) = \frac{x_n^2}{\alpha_n} \int_{\mathbf{R}^{n-1}} \frac{f(y')}{\|x - (y', 0)\|^n} dy' + \frac{n}{\alpha_n} x_n^3 \int_{\mathbf{R}^{n-1}} \frac{b}{\|x - (y', 0)\|^{n+2}} dy',$$

therefore

$$\theta - \lim_{x \rightarrow \xi} \frac{\partial}{\partial x_n} P(f, b, \mathbf{R}^{n-1})(x) = b.$$

$$I = \theta - \lim_{x \rightarrow \xi} \frac{\partial}{\partial x_n} P(\mu, \eta, \mathbf{R}^{n-1})(x) - \theta - \lim_{x \rightarrow \xi} \frac{\partial}{\partial x_n} P(f, b, \mathbf{R}^{n-1})(x),$$

then

$$\begin{aligned}
 (6) \quad I &= \theta - \lim_{x \rightarrow \xi} \left[\frac{2}{\alpha_n} x_n \int_{R^{n-1}} \left(\frac{1}{\|x - (y', 0)\|^n} d\mu(y') \right. \right. \\
 &\quad \left. \left. - b \int_{R^{n-1}} \frac{1}{\|x - (y', 0)\|^n} dy' \right) \right] - \theta - \lim_{x \rightarrow \xi} \left[\frac{n}{\alpha_n} x_n^3 \left(\int_{R^{n-1}} \frac{1}{\|x - (y', 0)\|^{n+2}} d\mu(y') \right. \right. \\
 &\quad \left. \left. - b \int_{R^{n-1}} \frac{1}{\|x - (y', 0)\|^{n+2}} dy' \right) \right] + \theta - \lim_{x \rightarrow \xi} \left[\frac{3n}{\alpha_n} x_n^2 \left(\int_{R^{n-1}} \frac{1}{\|x - (y', 0)\|^{n+2}} d\eta(y') \right. \right. \\
 &\quad \left. \left. - \int_{R^{n-1}} \frac{f(y')}{\|x - (y', 0)\|^{n+2}} dy' \right) \right] - \theta - \lim_{x \rightarrow \xi} \left[\frac{n(n+2)}{\alpha_n} x_n^4 \left(\int_{R^{n-1}} \frac{1}{\|x - (y', 0)\|^{n+4}} d\eta(y') \right. \right. \\
 &\quad \left. \left. - \int_{R^{n-1}} \frac{f(y')}{\|x - (y', 0)\|^{n+4}} dy' \right) \right].
 \end{aligned}$$

Let r_0 denote a positive number such that

$$|(K_{\xi, r})| < C_2 r^{\beta_2} \quad \text{for } r > r_0.$$

By the condition D we have

$$\int_{K_{\xi, r_0}} \frac{1}{\|x - (y', 0)\|^{n+2}} d\lambda(y') = \int_{K_{\xi, r_0}} \frac{1}{\|x - (y', 0)\|^{n+2}} d\eta(y') - \int_{K_{\xi, r_0}} \frac{1}{\|x - (y', 0)\|^{n+2}} f(y') dy'.$$

For the measure δ it is

$$\int_{K_{\xi, r_0}} \frac{1}{\|x - (y', 0)\|^n} d\delta(y') = \int_{K_{\xi, r_0}} \frac{1}{\|x - (y', 0)\|^n} d\mu(y') - b \int_{K_{\xi, r_0}} \frac{1}{\|x - (y', 0)\|^n} dy'.$$

Hence and from (6) we have

$$I = I_1 + I_2,$$

where

$$I_1 := \theta - \lim_{x \rightarrow \xi} \frac{\partial}{\partial x_n} P(\delta, \lambda, K_{\xi, r_0})(x),$$

$$I_2 := \theta - \lim_{x \rightarrow \xi} \frac{\partial}{\partial x_n} P(\mu, \eta, R^{n-1} \setminus K_{\xi, r_0})(x) - \theta - \lim_{x \rightarrow \xi} \frac{\partial}{\partial x_n} P(f, b, R^{n-1} \setminus K_{\xi, r_0})(x).$$

But

$$|I_1| \leq \theta - \lim_{x \rightarrow \xi} \left(\frac{2}{\alpha_n} x_n \int_{K_{\xi, r_0}} \frac{1}{\|x - (y', 0)\|^n} d|\delta|(y') \right)$$

$$\begin{aligned}
& + \theta - \lim_{x \rightarrow \xi} \left(\frac{n}{\alpha_n} x_n^3 \int_{K_{\xi', r_0}} \frac{1}{\|x - (y', 0)\|^{n+2}} d|\delta|(y') \right) \\
& + \theta - \lim_{x \rightarrow \xi} \left(\frac{3n}{\alpha_n} x_n^2 \int_{K_{\xi', r_0}} \frac{1}{\|x - (y', 0)\|^{n+2}} d|\lambda|(y') \right) \\
& + \theta - \lim_{x \rightarrow \xi} \frac{n(n+2)}{\alpha_n} x_n^4 \int_{K_{\xi', r_0}} \frac{1}{\|x - (y', 0)\|^{n+4}} d|\lambda|(y') \\
& = \theta - \lim_{x \rightarrow \xi} \frac{\partial}{\partial x_n} P(|\delta|, |\lambda|, K_{\xi', r_0})(x) \\
& \leq \theta - \lim_{x \rightarrow \xi} \frac{\partial}{\partial x_n} P(|\delta|, |\lambda|, \mathbf{R}^{n-1})(x).
\end{aligned}$$

Hence and from Lemma 3 $I_1 = 0$.

In order to prove that $I_2 = 0$ it is sufficient to check that

$$\begin{aligned}
(7) \quad & \theta - \lim_{x \rightarrow \xi} \left[\frac{2}{\alpha_n} x_n \left(\int_{\mathbf{R}^{n-1} \setminus K_{\xi', r_0}} \frac{1}{\|x - (y', 0)\|^n} d\mu(y') - b \int_{\mathbf{R}^{n-1} \setminus K_{\xi', r_0}} \frac{1}{\|x - (y', 0)\|^n} dy' \right) \right] = 0, \\
& \theta - \lim_{x \rightarrow \xi} \left[\frac{n}{\alpha_n} x_n^3 \left(\int_{\mathbf{R}^{n-1} \setminus K_{\xi', r_0}} \frac{1}{\|x - (y', 0)\|^{n+2}} d\mu(y') \right. \right. \\
& \quad \left. \left. - b \int_{\mathbf{R}^{n-1} \setminus K_{\xi', r_0}} \frac{1}{\|x - (y', 0)\|^{n+2}} dy' \right) \right] = 0, \\
& \theta - \lim_{x \rightarrow \xi} \left[\frac{3n}{\alpha_n} x_n^2 \left(\int_{\mathbf{R}^{n-1} \setminus K_{\xi', r_0}} \frac{1}{\|x - (y', 0)\|^{n+2}} d\eta(y') \right. \right. \\
& \quad \left. \left. - \int_{\mathbf{R}^{n-1} \setminus K_{\xi', r_0}} \frac{f(y')}{\|x - (y', 0)\|^{n+2}} dy' \right) \right] = 0, \\
& \theta - \lim_{x \rightarrow \xi} \left[\frac{n(n+2)}{\alpha_n} x_n^4 \left(\int_{\mathbf{R}^{n-1} \setminus K_{\xi', r_0}} \frac{1}{\|x - (y', 0)\|^{n+4}} d\eta(y') \right. \right. \\
& \quad \left. \left. - \int_{\mathbf{R}^{n-1} \setminus K_{\xi', r_0}} \frac{f(y')}{\|x - (y', 0)\|^{n+4}} dy' \right) \right] = 0.
\end{aligned}$$

These equalities are true, because the integrals in these equalities are bounded by a constant, independent of the point x sufficiently near to the point $(\xi', 0)$.

LEMMA 5. If f is absolutely integrable with respect to the Lebesgue measure on \mathbf{R}^{n-1} , then

$$(a) \quad \theta\text{-}\lim_{x \rightarrow \xi} \left(\frac{x_n}{\alpha_n} \int_{\mathbf{R}^{n-1}} \frac{f(y')}{\|x - (y', 0)\|^n} dy' \right) = f(\xi') \quad a.e.$$

and

$$(b) \quad \theta\text{-}\lim_{x \rightarrow \xi} \left(\frac{n}{\alpha_n} x_n^3 \int_{\mathbf{R}^{n-1}} \frac{f(y')}{\|x - (y', 0)\|^{n+2}} dy' \right) = f(\xi') \quad a.e.$$

Proof. (a) is a consequence of Theorem 1, Chapter VII of [7]. We shall prove (b). In virtue of

$$\int_{\mathbf{R}^{n-1}} \frac{1}{(s_1^2 + \dots + s_n^2 + 1)^{n+2/2}} ds_1 \dots ds_{n-1} = \frac{\alpha_n}{n}$$

(see [1], page 228), applying the change of variables $z_i - k_i = x_n s_i$ ($i = 1, \dots, n-1$) we have, for arbitrary k_i ($i = 1, \dots, n-1$)

$$(8) \quad \int_{\mathbf{R}^{n-1}} \frac{x_n^3}{(x_n^2 + \|z - k\|^2)^{n+2/2}} dz = \frac{\alpha_n}{n},$$

$z = (z_1, \dots, z_{n-1})$, $k = (k_1, \dots, k_{n-1})$, $dz = dz_1 \dots dz_{n-1}$. We have now

$$\begin{aligned} I &:= \frac{n}{\alpha_n} x_n^3 \int_{\mathbf{R}^{n-1}} \frac{f(y')}{\|x - (y', 0)\|^{n+2}} dy' - f(\xi') \\ &= \frac{n}{\alpha_n} x_n^3 \int_{\mathbf{R}^{n-1}} \frac{f(\xi' - z)}{(x_n^2 + \|z - (\xi' - x')\|^2)^{n+2/2}} dz - f(\xi') \end{aligned}$$

and because of (8) we have

$$\begin{aligned} I &= \frac{n}{\alpha_n} x_n^3 \int_{\mathbf{R}^{n-1}} \frac{f(\xi' - z)}{(x_n^2 + \|z - (\xi' - x')\|^2)^{n+2/2}} dz \\ &\quad - \frac{n}{\alpha_n} x_n^3 \int_{\mathbf{R}^{n-1}} \frac{f(\xi')}{(x_n^2 + \|z - (\xi' - x')\|^2)^{n+2/2}} dz \\ &= \frac{n}{\alpha_n} \int_{\mathbf{R}^{n-1}} \frac{x_n^3 [f(\xi' - z) - f(\xi')]}{(x_n^2 + \|z - (\xi' - x')\|^2)^{n+2/2}} dz. \end{aligned}$$

It is easy to check that for a fixed α , $0 \leq \alpha < \frac{\pi}{2}$, there exists a constant A_α such that

$$x_n^2 + \|z - (\xi' - x')\|^2 \geq A_\alpha (x_n^2 + \|z\|^2)$$

for arbitrary $z \in \mathbf{R}^{n-1}$ and $\|\xi' - x'\| \leq ax_n$, where $a = \operatorname{tg} \alpha$. We have

$$|I| \leq \frac{n}{\alpha_n} A_\alpha \int_{\mathbf{R}^{n-1}} \frac{x_n^3}{(x_n^2 + \|z\|^2)^{n+2/2}} |f(\xi' - z) - f(\xi')| dz$$

for $x_n > 0$.

Let ξ' be an arbitrary fixed Lebesgue-point of the function f (a set of such points have full Lebesgue measure, because f is integrable; see [7], § 1, Chapter 1).

Let us fix an arbitrary $\varepsilon > 0$. There is $\delta > 0$ such that

$$\frac{1}{\mathcal{O}_{n-1}(K_{\xi', r})} \int_{K_{\xi', r}} |f(y') - f(\xi')| dy' < \frac{\varepsilon}{2A_\alpha}$$

for $0 < r < \delta$.

Let

$$h(y') := \begin{cases} f(y') - f(\xi') & \text{for } \|y' - \xi'\| \leq \delta, \\ 0 & \text{for } \|y' - \xi'\| > \delta. \end{cases}$$

Then we have

$$(9) \quad (Mh)(\xi') := \sup_{r > 0} \left(\frac{1}{\mathcal{O}_{n-1}(K_{\xi', r})} \int_{K_{\xi', r}} |h(y')| dy' \right) < \frac{\varepsilon}{2A_\alpha}.$$

Let us denote

$$\varphi(z) := \frac{x_n^3}{(x_n^2 + \|z\|^2)^{n+2/2}}$$

and

$$\varphi_\varepsilon(z) := \varepsilon^{-(n-1)} \varphi\left(\frac{z}{\varepsilon}\right)$$

for $\varepsilon > 0$. We have (see [7], Chapter III, Theorem 2a)

$$\sup_{\varepsilon > 0} \left| \int_{\mathbf{R}^{n-1}} \varphi_\varepsilon(z) |h(\xi' - z)| dz \right| \leq \frac{\alpha_n}{n} (Mh)(\xi').$$

In particular, for $\varepsilon = 1$ we have

$$\left| \int_{\mathbf{R}^{n-1}} \varphi_1(z) |h(\xi' - z)| dz \right| \leq \frac{\alpha_n}{n} (Mh)(\xi')$$

i.e.

$$\int_{\mathbf{R}^{n-1}} \frac{x_n^3}{(x_n^2 + \|z\|^2)^{n+2/2}} |h(\xi' - z)| dz \leq \frac{\alpha_n}{n} (Mh)(\xi').$$

