UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXVI

1987

Differentiability of Solutions with Respect to Parameters

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1. Introduction. Assuming that X is a Hilbert space we let $B(X, Y)(B(X) = B(X, X))$ be the vectorspace of all linear, bounded operators $A\colon X \to Y$. If $A\colon X \to Y$ is a linear operator, then $D(A)N(A)$, $R(A)$, $[A]$ will denote the domain, kernel, range and closure A, respectively. The identity operator will be denoted by I. If Z is a linear subspace of a Hilbert space X then the orthogonal completion of Z will be denoted by $Z^{\perp}$.

Let Ω be an open subset of R^m. Let $\{A_h\}$ (with $h \in \Omega$) be a family of linear operators $A_h\colon X \to Y$ with a dense domain $D(A_h)$. We consider the family of equations of the form

$$A_h u_h = f_h, \quad \text{for } h \in \Omega, \tag{1}$$

where $f_h \in Y$ is given and $u_h \in X$ is an unknown point of $D(A_h)$.

In this note we shall study k times differentiability of the mapping

$$\Omega \ni h \to u_h \in X, \tag{2}$$

where u_h is a solution of (1). We shall use the following assumptions:

ASSUMPTION Z_1: $\dim N_h = \alpha < \infty$, $D(A_h) = D$, *where* $N_h = \ker A_h$.
For fixed $a \in \Omega$.

ASSUMPTION Z_2(a): *there exist* $T_a \in B(Y, X)$ *such that* $T_a A_a = I - P_a$, *where* P_a *is the operator of orthogonal projection of* X *onto* N_a.

ASSUMPTION Z_3(a): *there exists an open neighborhood* U_a *of the point* a *such that for every* $h \in U_a$ *the operator* $[T_a A_h]$ *is bounded and the mapping*

$$U_a \ni h \to [T_a A_h] \in B(X)$$

is differentiable at a, *i.e. there exists a bounded linear operator* $W_a \in B(R^m\colon B(X))$ *such that*

$$|h|^{-1}\|[T_a A_{a+h} - T_a A_a] - W_a h\| \to 0 \text{ when } |h| \to 0 \tag{3}$$

ASSUMPTION Z_4(a): *the mapping*

$$\Omega \ni h \to T_a f_h \in X \tag{4}$$

is differentiable at the point a. *We shall denote by* B_a *the differential of the mapping* (4) *at* a.

M. Schechter proved in [2] that if for every $a \in \Omega$ the assumptions Z_1, Z_2(a), Z_3(a), Z_4(a) hold and $f_h \in R(A_h)$ for $h \in \Omega$, then the mapping (2) is differentiable in Ω. By applying this result to the family $\{A_h\}$, $h \in \Omega$ of elliptic operators with all coefficients of C^∞ class with respect to both x and h it has been proved that the solution $u_h \in N_h^\perp$ is differentiable in Ω, with respect to h. Next step of Schechter's proof (presented in [2]) of second order differentiability of the maping $h \to u_h$ is supported on the incorrect fact that $\frac{\partial u_h}{\partial h_j} \in N_h^\perp$ ([2], p. 607).

Another approach to the proof makes it possible for us to prove the theorem on infinite differentiability of the mapping (2) with respect to h.

2. Helpful Lemmas. We begin with two lemmas that are not closely connected with equation (1).

Let, us previously, be a Hilbert space with the scalar product $<,>$. Let U be an open subset of $\boldsymbol{R}^m$. Let $\{\{X_h\}$, for $h \in U$, be a family of α-dimensional linear subspaces of X $(1 \leqslant \alpha < \infty)$. Let $e_1(h), \ldots, e_\alpha(h)$ be a base of X_h for $h \in U$. Then every vector $x \in X$ can be uniquely presented in the form

$$x = \xi(x, h) + \eta(x, h),$$

where $\xi(x, h) \in X_h$, $\eta(x, h) \in X_h^\perp$ for $h \in U$.

Lemma 1. *If $x_h \in X$ for $h \in U$ and the mapping*

$$U \ni h \to x_h \in X$$

is l times differentiable at a point $a \in U$ and mappings

$$U \ni h \to e_j(h) \in X, \quad j = 1, \ldots, \alpha$$

are at least l times differentiable at a, then the mappings

$$U \ni h \to w_h = \eta(x_h, h) \in X_h^\perp \in X$$

and

$$U \ni h \to v_h = \xi(x_h, h) \in X_h \subset X$$

are l times differentiable at a.

Proof. Since $e_1(h), \ldots, e_\alpha(h)$ is a base of X_h, there exist real functions $\delta_1, \ldots, \delta_\alpha \colon U \to \boldsymbol{R}$ such that

$$v_h = \delta_1(h)e_1(h) + \ldots + \delta_\alpha(h)e_\alpha(h).$$

Hence

$$x_h = \delta_1(h)e_1(h) + \ldots + \delta_\alpha(h)e_\alpha(h) + w_h$$

Multiplying scalarly both sides by $e_j(h)$, $j = 1, \ldots, \alpha$ we obtain

(5) $$<x_h, e_j(h)> = \delta_1(h)<e_1(h), e_j(h)> + \ldots + \delta_\alpha(h)<e_\alpha(h)e_j(h)> \quad j = 1, \ldots, \alpha$$

The determinant

$$A(h) = \det<e_i(h), e_j(h)>, \quad i, j = 1, \ldots, \alpha$$

of coefficients of the system of equations (5) with unknown quantities $\delta_1(h), \ldots, \delta_\alpha(h)$ is different from 0, for $h \in U$ (because it is the Gramm determinant) and the mappings

$$U \ni h \to \langle x_h, e_j(h) \rangle, \qquad U \ni h \to A(h)$$

are l times differentiable at a. Hence, and from Cramer's formulas we obtain l times differentiability of δ_j at a, for $j = 1, \ldots, \alpha$. Therefore the mappings $h \to v_h$ and $h \to w_h$ are l times differentiable at a.

Let, as previously, a be a point of U and suppose that

$$X_a^\perp \cap X_h = \{0\} \quad \text{for } h \in U \tag{6}$$

Since the dimension of any X_h is equal to $\alpha < \infty$, the condition (6) implies that $X = X_h \oplus X_a^\perp$ for $h \in U$, where $\oplus$ denotes the direct sum. Then every $x \in X$ may be uniquely presented in the form

$$x = \xi_a(x, h) + \eta_a(x, h)$$

with $\xi_a(x, h) \in X_h$ and $\eta_a(x, h) \in X_a^\perp$.

LEMMA 2. *If $x_h \in X_h^\perp$ for $h \in U$ and the mapping*

$$U \ni h \to x_h \in X$$

is l times differentiable ($l \geq 0$) and

$$U \ni h \to e_j(h), \quad j = 1, \ldots, \alpha$$

are at least l times differentiable in U, then there exists a neighborhood $V \subset U$ of a such that the mappings

$$V \ni h \to \eta_a(x_h, h) \in X_a^\perp$$

and

$$V \ni h \to \xi_a(x_h, h) \in X_h$$

are l times differentiable

Proof. Since $e_1(h), \ldots, e_\alpha(h)$ is a base of X_h, there exist functions $\beta_1, \ldots, \beta_\alpha$: $U \to \mathbf{R}$ such that

$$\xi_a(x_h, h) = \beta_1(h)e_1(h) + \ldots + \beta_\alpha(h)e_\alpha(h).$$

Therefore

$$x_h = \beta_1(h)e_1(h) + \ldots + \beta_\alpha(h)e_\alpha(h) + \eta_a(x_h, h) \tag{7}$$

Multiplying both sides of (7) by $e_j = e_j(a)$ we obtain

$$\langle x_h, e_j \rangle = \beta_1(h)\langle e_1(h), e_j \rangle + \ldots + \beta_\alpha(h)\langle e_\alpha(h), e_j \rangle, \quad j = 1, \ldots, \alpha \tag{8}$$

The determinant $B(h) = \det(\langle e_i(h), e_j(a) \rangle)$ is of the same class as functions $h \to e_j(h)$ and $B(a) \neq 0$. Hence there exists a neighborhood $V \subset U$ of a such that $B(h) \neq 0$ for $h \in V$. Now, using Cramer's formulas we prove l times differentiability of the mapping

$$V \ni h \to \xi_a(x_h, h).$$

Since $\eta_a(x_h, h) = x_h - \xi_a(x_h, h)$ the, mapping

$$V \ni h \to \eta_a(x_h, h)$$

is l times differentiable too.

LEMMA 3. *If the assumption $Z_1, Z_2(a), Z_3(a)$ hold then there exists a neighborhood U_1 of a such that $N_a^\perp \cap N_h = \{0\}$ every $h \in U_1$.*

Proof. Let $S = \{x \in X: ||x|| = 1\}$. It follows from Assumption Z_3 that the mapping

$$U_a \ni h \to [T_a A_h]$$

is differentiable at a. Then

$$[T_a A_h]x \to [T_a A_a]x, \quad \text{when } h \to a,$$

uniformly on S.
But $||[T_a A_a]x|| = 1$ for $x \in S \cap N_a^\perp$. Hence

$$||[T_a A_h]x|| \to 1, \quad \text{when } h \to a,$$

uniformly on $S \cap N_a^\perp$. Therefore, there exists a neighborhood $U_1 \subset U_a$ of a such that

(9) $$||[T_a A_h]x|| \geqslant \tfrac{1}{2} \quad \text{for } h \in U_1$$

Let $x \in N_a^\perp - \{0\}$. Then $\frac{x}{||x||} \in S \cap N_a^\perp$ and (9) implies

(10) $$||[T_a A_h]x|| \geqslant \tfrac{1}{2}||x|| \quad \text{for } x \in N_a^\perp,\ h \in U_1$$

Hence

$$N_a^\perp \cap N_h = \{0\} \quad \text{for } h \in U_1.$$

COROLLARY 1. *There exists an open neighborhood U_1 of a in $\mathbf{R}^m$ such that X is the direct sum of its subspaces N_h and $N_a^\perp$ for $h \in U_1$.*

LEMMA 4. *Let $Z_1, Z_2(a), Z_3(a)$. If $v \in D$ and f_h is defined by $f_h = A_h v$, then there exists a bounded linear operator B: $\mathbf{R}^m \to X$ such that*

(11) $$|h|^{-1}||T_a(f_{a+h} - f_a) - Bh|| \to 0, \quad \textit{when } h \to 0.$$

Proof. It suffices to take $Bh = (W_a h)v$.

Differentiability with respect to parametr.

THEOREM 1. *Suppose that $Z_1, Z_2(a), Z_3(a), Z_4(a)$ hold, let $u_h \in D \cap N_a^\perp$ and let*

(12) $$A_h u_h = f_h \quad \textit{for } h \in U_1.$$

Then u_h is a unique solution of (12) (for $h \in U_1$) and the mapping

$$U_1 \ni h \to u_h \in X$$

is differentiable at a.

Proof. The uniqueness follows immediately from the fact that $N_a^\perp \cap N_h = \{0\}$ for $h \in U_1$. Let $u_h \in N_a^\perp$ be this unique solution of (12) for $h \in U_1$. According to (10) we infer that

$$\|u_h\| \leqslant 2\|T_a A_h u_h\| = 2\|T_a f_h\| \leqslant C_1$$

and conclude

$$\|u_h\| \leqslant C_1 \quad \text{for } h \in U_1 \tag{13}$$

Since $u_h \in N_a^\perp$ for $h \in U_1$ and $N_a^\perp$ is a linear subspace of X, we conclude that $u_h - u_a \in N_a^\perp$ and

$$\begin{aligned} u_h - u_a = T_a A_a(u_h - u_a) &= T_a A_a u_h - T_a A_h u_h + T_a A_h u_a - T_a A_a u_a \\ &= T_a(A_a - A_h)u_h + T_a(f_h - f_a). \end{aligned} \tag{14}$$

Assumption $Z_3(a)$, $Z_4(a)$ and (13), (14) imply

$$u_h - u_a \to 0 \quad \text{as } h \to a$$

Moreover, denoting $h = a + \Delta h$, where Δh belongs to a neighborhood of 0 in $\mathbf{R}^m$ we find that

$$\begin{aligned} |\Delta h|^{-1}\|u_{a+\Delta h} - u_a + (W_a \Delta h)u_{a+\Delta h} - B_a \Delta h\| &\leqslant |\Delta h|^{-1}\|T_a(A_a - A_{a+\Delta h} \\ - W_a \Delta h\|\ \|u_{a+\Delta h}\| + |\Delta h|^{-1}\|T_a(f_{a+\Delta h} - f_a) - B_a \Delta h &\to 0 \quad \text{as} \quad \Delta h \to 0. \end{aligned}$$

Since

$$|\Delta h|^{-1}\|(W_a \Delta h)(u_{a+\Delta h} - u_a)\| \leqslant \|W_a\|\,\|u_{a+\Delta h} - u_a\| \to 0 \quad \text{as } \Delta h \to 0,$$

we conclude that

$$\begin{aligned} |\Delta h|^{-1}\|u_{a+\Delta h} - u_a + (W_a \Delta h)u_a - B_a \Delta h\| &= |\Delta h|^{-1}\|T_a(A_a - A_{a+\Delta h}) + W_a \Delta h\|\,\|u_{a+\Delta h}\| \\ + |\Delta h|^{-1}\|T_a(f_{a+\Delta h} - f_a) - B_a \Delta h\| + |\Delta h|^{-1}\|W_a \Delta h(u_{a+\Delta h} - u_a)\| &\to 0 \quad \text{as } \Delta h \to 0 \end{aligned}$$

Hence the operator $E \in B(\mathbf{R}^m : X)$ defined by

$$E\Delta h = -(W_a \Delta h)u_0 + B_a \Delta h$$

is the differential of the mapping $h \to u_h$ at the point a. Similarly we can prove

THEOREM 1a. *Suppose that* Z_1, $Z_2(a)$, $Z_3(a)$ *hold. If* $u_h \in D \cap N_a^\perp$ *is such that the mapping*

$$U_1 \ni h \to T_a A_h u_h \in X$$

is differentiable at a, then the mapping

$$U_1 \ni h \to u_h \in X$$

is differentiable at a.

THEOREM 2. *Suppose* Z_1, $Z_2(a)$, $Z_3(a)$, $Z_4(a)$. *Then there exist a sufficiently small neighborhood* U_2 *of the point a and a base* $e_1(h), \ldots, e_\alpha(h)$ *of* N_h, *for* $h \in U_2$ *such that the* *mappings*

$$U_2 \ni h \to e_j(h) \in X, \quad j = 1, \ldots, \alpha$$

are differentiable at a.

Proof. Let $e_1, \ldots, e_\alpha$ be a basis of the subspace N_a. Let U_2 be a neighborhood of a such as in Lemma 3. Then $X = N_h \oplus N_a^\perp$ for $h \in U_2$ and

$$e_j = a_j(h) + b_j(h), \tag{15}$$

where $a_j(h) \in N_h$, $b_j(h) \in N_a^\perp$ are unique.
Hence

$$A_h e_j = A_h(b_j(h)) \tag{16}$$

and by Lemma 4 the function $h \to f_h = A_h e_j$ satisfies Assumption Z_4(a). Moreover, it follows from Theorem 1 that the mapping

$$U_2 \ni h \to b_j(h), \quad j = 1, \ldots, \alpha$$

are differentiable at a. Letting

$$e_j(h) = e_j - b_j(h), \quad j = 1, \ldots, \alpha \tag{17}$$

we see that the mappings

$$U_2 \ni h \to e_j(h) \in X, \quad j = 1, \ldots, \alpha$$

are differentiable at a. Now, it suffices to show that $e_1(h), \ldots, e_\alpha(h)$ are linearly independent. If

$$\lambda_1 e_1(h) + \ldots + \lambda_\alpha e_\alpha(h) = 0, \tag{18}$$

then $\lambda_1(e_1 - b_1(h)) + \ldots + \lambda_\alpha(e_\alpha - b_\alpha(h)) = 0$
yields

$$\lambda_1 e_1 + \ldots + \lambda_\alpha e_\alpha - \lambda_1 b_1(h) - \lambda_2 b_2(h) - \ldots - \lambda_\alpha b_\alpha(h) = 0. \tag{19}$$

Since $X = N_a \oplus N_a^\perp$, we see from (19) that

$$\lambda_1 e_1 + \ldots + \lambda_\alpha e_\alpha = 0 \tag{20}$$

Hence $\lambda_1 = \lambda_2 = \ldots = \lambda_\alpha = 0$, because $e_1, \ldots, e_\alpha$ are linearly independent.

Let $u_h \in N_a^\perp \subset X$ be a solution of the equation $A_h u_h = f_h$. Since $X = N_h \oplus N_h^\perp$, so $u_h = w_h + v_h$, where $w_h \in N_h$, $v_h \in N_h^\perp$. Thus $A_h v_h = f_h$.

THEOREM 3. *If $v_h \in N_h^\perp$ is such that $A_h v_h = f_h$ and assumptions Z_1, Z_2(a), Z_3(a), Z_4(a) hold then the mapping $h \to v_h \in N_h^\perp \subset X$ is differentiable at a.*

Proof. It is a simple consequence of Theorem 2 and Lemma 1.

Suppose that for every $a \in \Omega$ assumptions Z_1, Z_2(a), Z_3(a), Z_4(a) hold. Then it follows from Theorem 3 that the mapping $\Omega \ni h \to v_h \in N_h^\perp \subset X$ is differentiable at every point of Ω. This fact permits us to prove

THEOREM 4. *If assumptions* Z_1, $Z_2(a)$, $Z_3(a)$ *are fulfilled at every point* $a \in \Omega$, *then for every* $a \in \Omega$ *there exists a neighborhood* $U_3 \subset U_a$ *of the point* a *and a base* $e_1(h), \ldots, e_\alpha(h)$ *of* N_h *for* $h \in U_3$ *such that the mappings*

$$U_a \ni h \to e_j(h) \in X \qquad j = 1, \ldots, \alpha$$

are differentiable.

Proof. Let $a \in \Omega$ and let $U_3 \subset U_a$ be a neighborhood of a such that $N_h \cap N_a^\perp = \{0\}$ for $h \in U_3$ (it follows from Lemma 3). Let $e_1, \ldots, e_\alpha$ be a base of N_a. Since $X = N_h \oplus N_h^\perp$, there exist (exactly one) $e_j(h) \in N_h$, $b_j(h) \in N_h^\perp$ such that

$$e_j = e_j(h) + b_j(h) \quad \text{for } h \in U_3 .$$

We see that

$$A_h e_j = A_h(b_j(h))$$

and the proof of Theorem 4 is similar to the proof of Theorem 2. Combining this theorem with Lemma 2 we obtain

COROLLARY 2. *If* U_a *is a such neighborhood of* a *as in Theorem* 4, $f_h \in R(A_h)$ *for* $h \in \Omega$ *and Assumption* $Z_4(a)$ *hold then for every* $h \in U_a$ *there exists a solution* u_h *of the equation* $A_h u_h = f_h$ *such that*

a) $u_h \in N_a^\perp$ *for* $h \in U_a$

b) *there exists a neighborhood* $V \subset U_a$ *of* a *such that the mapping*

$$V \ni h \to u_h \in N_a^\perp \subset X$$

in differentiable in V.

Suppose that $v_h \in X$ fulfils equation $A_h v_h = f_h$ for $h \in \Omega$ and suppose that differentiations A_h', v_h', f_h' (with respect to h) are sensible. Then $A_h v_h' = f_h' - A_h' v_h = \tilde{f}_h$. Therefore v_h' is a solution of the equation of the same type as equation (1). In this case it may happen $v_h \in N_h^\perp$ and $v_h' \notin N_h^\perp$. Then the theorem on differentiability with respect to parameter of $v_h \in N_h$ does not make it possible for us to prove the theorem on 2 times differentiability of v_h (cp. [2], p. 607).

We shall present an approach which makes it possible to get the following three theorems on higher order differentiability of v_h.

THEOREM 5. *If assumptions* Z_1, $Z_2(a)$ *and* $Z_3(a)$ *are fulfilled at every point* $a \in \Omega$, $f_h \in R(A_h)$ *for* $h \in \Omega$ *and for every* $a \in \Omega$ *the mappings*

$$\Omega \ni h \to T_a f_h \in X \quad \text{and} \quad U_a \ni h \to [T_a A_h] \in B(X)$$

are l *times differentiable at* $(l \geq 1)$ *then*

1_0 *for every* $a \in \Omega$ *there exists a neighborhood* U_a *of* a *such that* $N_h \cap N_a^\perp = \{0\}$ *for* $h \in U_a$,

2_0 *for every* $h \in U_a$ *there exists exactly one* $u_h \in N_a^\perp$ *such that* $A_h u_h = f_h$, 3$_0$ *for every* $a \in \Omega$ *there exists a neighborhood* $V_a \subset U_a$ *of* a *(independent of* l*) such that the mapping*

$$V_a \in h \to u_h \in X$$

is l *times differentiable in* V_a.

THEOREM 6. *Under all the assumptions of Theorem 5, for every* $a \in \Omega$ *there exists an open neighborhood* U_a *of* a *and a base* $e_1(h), \ldots, e_\alpha(h)$ *of* N_h *for* $h \in \Omega$ *such that the mappings*

$$U_a \ni h \to e_j(h) \in X, \quad j = 1, \ldots, \alpha$$

are l *times differentiable in* U_a.

THEOREM 7. *Under all the assumptions of Theorem 5, if* $v_h \in N_h^\perp$ *is such that* $A_h v_h = f_h$ *for* $h \in \Omega$*, then the mapping*

$$\Omega \ni h \to v_h \in X$$

is l *times differentiable in* Ω.

Proof of Theorems 5.6.7. Suppose that Theorem 5 holds. Then reasoning similar to that in the proof of Theorem 3 gives Theorem 7. Similarly, we obtain Theorem 6 by using the reasoning of the proof of Theorem 4.

If $l = 1$ then, by Corollary 2, Theorem 5 is true. According to Corollary 2, the mapping

$$V_a \ni h \to u_h \in N_a^\perp \subset X$$

is differentiable. Moreover, we have

$$T_a A_h u_h = T_a f_h, \quad \text{for } h \in V_a \tag{21}$$

Differentiating both sides of (21) with respect to h we obtain

$$(T_a A_h) u_h' = (T_a f_h)' - (T_a A_h)' u_h .$$

If $l \geqslant 2$, then we observe that $u_h' \in N_a^\perp$ and

$$T_a A_h u_h' = F_h \quad \text{for } h \in V_a,$$

with $F_h = (T_a f_h)' - (T_a A_h)' u_h$ and it follows from Theorem 1a that $h \to u_h'$ is differentiable in an open neighborhood of a. Now continuation of inductive reasoning is clear.

Now we recall an example. Let as in [2], G be a bounded domain in $\boldsymbol{R}^n$ with G^∞ boundary ∂G. Let

$$A_h = \sum_{|\mu| \leqslant m} a_\mu(x, h) D^\mu$$

be appropely elliptic operator of order m with coefficients a_μ of class C^∞ in $\bar{G} \times \Omega$, where Ω is an open subset of $\boldsymbol{R}^k$. Let

$$B_j = \sum_{|\mu| \leqslant m_j} b_{j\mu}(x) D^\mu, \quad 1 \leqslant j \leqslant \frac{m}{2}$$