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Periods of Slowly Oscillating Solutions of First Order Delay Differential Equations

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1. Introduction. This paper deals with slowly oscillating periodic solutions of the equation

$$\text{(E)} \qquad \dot{x}(t) = -f\big(x(t), x(t-1)\big).$$

A know example of such equation is the Ważewska-Lasota [3] equation of the red blood cells survival:

$$\text{(F)} \qquad \dot{u}(t) = -\sigma u(t) + e^{-u(t-h)},$$

where $u(t)$ denotes the amount of cells in the time t, and σ is the coefficient of destruction. In 1974 S. N Chow [1] proved the existence a nonconstant periodic solution of equation (F). For the proof, he used the theorem of Browder on nonejective fixed points. Later in 1977 J. L. Kaplan and J. A. Yorke [2] studied the behaviour of solutions of (E) in the $\big(x(t), \dot{x}(t)\big)$ phase plane by a geometrical method. They proved the existence of slowly oscillating periodic solutions of equation (E) (see Def. 1 below). They also estimated the amplitudes of such solutions. In this paper we shall estimate the period of slowly oscillating periodic solutions of equation (E) and we shall apply our general result to equation (F).

2. Results. Throughout the paper we assume that f is continuously differentiable and, at various stages, we shall require some or all of the following hypotheses on f:

$$\text{(1)} \qquad f(0,0) = 0$$

$$\text{(2)} \qquad (\partial f/\partial y)(x, y) > 0 \qquad \text{for all } (x, y) \in R^2$$

$$\text{(3)} \qquad N \geqslant (\partial f/\partial x)(x, y) \geqslant 0 \qquad \text{for all } (x, y) \in R^2 \quad \text{and for some } N$$

$$\text{(4)} \qquad f(0, y) > -B \qquad \text{for some } B > 0.$$

DEFINITION 1. We say that (E) is *oscillatorilly unstable* if its linearization

$$\text{(L)} \qquad \dot{x}(t) = -a_1 x(t) - a_2 x(t-1),$$

where $a_1 = (\partial f/\partial x)(0,0) \geqslant 0$ and $a_2 = (\partial f/\partial y)(0,0) > 0$ is unstable.

DEFINITION 2. A differentiable periodic function $x\colon R \to R$ with period $p>0$ is said to be *slowly oscillating* if it satisfies the following two conditions:

1° There are three consequtive zeros z_1, z_2, z_4 of x such that $z_2>z_1+1$, $z_3=z_1+p>z_2+1$ and such that $x(t)>0$ for $t\in(z_1, z_2)$ and $x(t)<0$ for $t\in(z_2, z_3)$.

2° If $x(t_1)\neq 0$ for some t_1 then t_1 is not a local minimum of $|x(t)|$.
From Theorem 2.7 in [2] follows

THEOREM 1 (Kaplan and Yorke). *Let $f\colon R\to R$ be a continuously differentiable function which satisfies the hypotheses* (1)—(4) *and let* (E) *be oscillatorilly unstable. Then there exists a slowly oscillating periodic solution of* (E).

We shall prove in section 3 the following theorems:

THEOREM 2. *Let x be a slowly oscillating periodic solution of* (E). *Assume that f satisfies* (1)—(3) *and that* (E) *is oscillatorilly unstable. Write*

$$M=\max_{t\in R} x(t)\,,\qquad m=\min_{t\in R} x(t)\,,\qquad f_2(x,y)=(\partial f/\partial y)(x,y)\,.$$

Then the period p of x satisfies the inequality

$$2\leqslant p\leqslant 4+\alpha\beta\,,$$

where $\alpha=\max\limits_{0\leqslant y\leqslant M} f_2(0,y)$, $\beta=\max\limits_{m\leqslant y\leqslant 0} f_2(0,y)$.

THEOREM 3. *Let $0<\sigma<1/e$ be fixed. Then for any sufficiently large $h>0$ there exists a nonconstant periodic solution of* (F). *Its period p satisfies*

$$2h\leqslant p\leqslant 4h+(h\sigma\gamma)^3 e^{-(1/\sigma e^{1/\sigma})}\,, \tag{5}$$

where γ is the positive real root of the equation $\sigma\gamma=e^{-\gamma}$.

3. Proofs. Proof of Theorem 2. Let z_1, z_2, z_3, z_4 be a sequence of consecutive zeros of the solution x. Set $p_1=z_2-z_1$ and $p_2=z_3-z_2$, then $p=p_1+p_2$. We may assume that the function x is positive in (z_1, z_2). We prove that $p_1\leqslant 2+\alpha\beta/2$. A necessary condition for unstability of (L) is $a_2\geqslant \dfrac{\pi}{2}$. This implies that $\alpha\geqslant\dfrac{\pi}{2}$ and $\beta\geqslant\dfrac{\pi}{2}$, and consequently we may assume that $p_1\geqslant 3$.

For $t\in(z_1+1, z_2)$ we have $x(t)>0$ and $x(t-1)>0$. By (1) (2) (3) we obtain $\dot{x}(t)<0$ for $t\in(z_1+1, z_2)$. Hence

$$\ddot{x}(t)=-f_1\big(x(t), x(t-1)\big)\cdot\dot{x}(t)-f_2\big(x(t), x(t-1)\big)\cdot\dot{x}(t-1)>0$$

for $t\in(z_1+2, z_2)$ (where $f_1(x,y)=(\partial f/\partial x)(x,y)$).
By the concavity of x in (z_1+2, z_2)

$$x(t)\leqslant x(z_1+2)+\frac{x(z_2)-x(z_1+2)}{z_2-(z_1+2)}\cdot[t-(z_1+2)]\leqslant M(z_2-t)/(p_1-2)\,. \tag{6}$$

We shall estimate x in $[z_2, z_3]$. For $t \in [z_2, z_2+1]$ we have $x(t) \leqslant 0$, hence

$$\dot{x}(t) = -f(x(t), x(t-1)) > -f(0, x(t-1)) \geqslant -\alpha x(t-1).$$

By (6) we get for $t \in [z_2, z_2+1]$

$$\dot{x}(t) \geqslant -\alpha M(z_2+1-t)/(p_1-2). \tag{7}$$

Write $\gamma = -\alpha M/(p_1-2)$. By (7) for each $t \in [z_2, z_2+1]$

$$x(t) = x(z_2) + \int_{z_2}^{t} \dot{x}(s)\,ds \geqslant \gamma/2.$$

Since x is increasing in $[z_2+1, z_3]$ we have $x(t) \geqslant \gamma/2$ in $[z_2, z_3]$. We shall estimate x in $[z_3, z_4]$. For $t \in [z_3, z_3+1]$, we obtain

$$x(t) = x(z_3) + \int_{z_3}^{t} \dot{x}(s)\,ds \leqslant -\beta\gamma(t-z_3)/2 \leqslant -\beta\gamma/2.$$

The solution x is decreasing in $[z_3+1, z_4]$ and x is periodic. This implies that $x(t) \leqslant -\beta\gamma/2$ for all $t \in R$. On the other hand there is a point $t_0 \in R$ with $x(t_0) = M$. Hence $M \leqslant -\beta\gamma/2$. This implies $p_1 \leqslant 2+\alpha\beta/2$. The proof that $p_2 \leqslant 2+\alpha\beta/2$ is analogous to the proof for p_1.

Proof of Theorem 3. The substitution $x(t) = u(th)-\gamma$ into (F) yields an equivalent equation

$$\dot{x}(t) = -\sigma h x(t) + h\sigma\gamma(e^{-x(t-1)}-1). \tag{8}$$

This is in the form (E) in which

$$f(x, y) = \sigma h x - h\sigma\gamma(e^{-y}-1). \tag{9}$$

For $\sigma > 0$ and $h > 0$, f satisfies the hypothesis (1)—(3). Let $0 < \sigma < 1/e$ be fixed. Then for any sufficiently large $h > 0$, (8) is oscillatorilly unstable (see [1] Lemma 5.3). The slowly oscillating solutions of (8) are bounded (see [1] Lemma 5.2), and consequently all the assumptions of Theorem 1 are satisfied (see [2] Theorem 3.5). Thus the equation (8) has a slowly oscillating periodic solution x.

In [3] it is show that every solution x of the equation (F) satisfies the inequalities

$$\frac{1}{\sigma} e^{-1/\sigma} < \liminf_{t\to\infty} x(t) \leqslant \limsup_{t\to\infty} x(t) < \frac{1}{\sigma}.$$

Thus every periodic solution x of the equation (8) satisfies

$$m = \frac{1}{\sigma} e^{-1/\sigma} - \gamma < x(t) < \frac{1}{\sigma} - \gamma = M. \tag{10}$$

For the equation (8) we have $f_2(0, y) = h\sigma\gamma e^{-y}$. By (10) we obtain

$$\alpha = \max_{0 \leqslant y \leqslant M} h\sigma\gamma e^{-y} = h\sigma\gamma$$

$$\beta = \max_{m \leqslant y \leqslant 0} h\sigma\gamma e^{-y} = h\sigma^2\gamma^2 e^{(-1/\sigma e^{1/\sigma})}.$$

By Theorem 2 there exists a nonconstant periodic solution of (8) of the period p with satisfies

$$2 \leqslant p \leqslant 4 + h^2 \sigma^3 \gamma^3 e^{(-1/\sigma e^{1/\sigma})}.$$

This implies the inequality (5).

References

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[2] J. L. Kaplan and J. A. Yorke, *On the nonlinear differential delay equation* $\dot{x}(t) = -f(x(t), x(t-1))$. J. Difierential Equations 23 (1977), 293—314.

[3] M. Ważewska-Czyżewska, A. Lasota, *Matematyczne problemy dynamiki układu krwinek czerwonych.* Mat. Stos. 6 (1976), 23—40.

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