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Ergodic Measures on Topological Spaces

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1. Introduction. The classical results concerning the existence of invariant measures on topological spaces are due to Oxtoby and Ulam [5]. However, the problem of the existence of continuous (vanishing at points) invariant measures is still difficult. Recently A. Lasota and G. Pianigiani have been studying this problem using the method of A. Avez [1]. The main result obtained by them is the existence of continuous invariant measures for transformations satisfying so called N-addicity condition (see formula [2]). In the special case $N = 2$ they also proved the existence of an ergodic continuous invariant measure. The same problem for $N > 2$ has been open. The purpose of this paper is to prove the existence of continuous ergodic invariant measures for transformations satisfying N-adicity condition with arbitrary N. Our proofs are partially based on the technique developed by Lasota and Yorke [3]. Moreover, we give some sufficient conditions for the exactness of the resulting dynamical system.

2. Preliminaries. Let X be a topological Hausdorff space and let T be a continuous mapping from X into itself. For any $x_0 \in X$ the set

$$\mathcal{O}(x_0) = \{x_0, T(x_0), T^2(x_0), \ldots\}$$

is called the trajectory starting x_0, and the set

$$L(x_0) = \bigcap_{n=0}^{\infty} \mathrm{cl}\big(\mathcal{O}(T^n(x_0))\big) \quad (\mathrm{cl} = \text{closure})$$

is called the limit set of $\mathcal{O}(x_0)$. To underline the role of the transformation T we shall write sometimes $\mathcal{O}_T(x_0)$ and $L_T(x_0)$ for the trajectory and the limit set respectively.

DEFINITION 1. A trajectory $\mathcal{O}(x_0)$ is called *strictly turbulent* if the following two conditions hold

(i) $L(x_0)$ is a compact nonempty set,

(ii) $L(x_0)$ does not contain periodic points.

PROPOSITION 1. *Let T: $X \to X$ be a continuous function and let $S = T^k$ for some integer $k \geqslant 1$. For each $x_0 \in X$ the following two conditions are equivalent*

(a) $\mathcal{O}_T(x_0)$ *is strictly turbulent for* T,

(b) $\mathcal{O}_S(x)$ *is strictly turbulent for* S *and* $L_T(x_0)$ *is compact.*

The proof follows immediately from the definition.

By a measure we mean any regular probabilistic measure defined on the σ-algebra of Borel subsets of X. The measure m is called invariant under T if $m(T^{-1}(E)) = m(E)$ for each Borel subset E of X. We say that m is continuous if $m(\{x\}) = 0$ for each singleton $\{x\}$. The measure m is called ergodic if for each Borel subset E the condition $E = T^{-1}(E)$ implies $m(E)(1-m(E)) = 0$.

PROPOSITION 2 (see [3]). *Let* $T\colon X \to X$ *be a continuous mapping and let* $\mathcal{O}(x_0)$ *be a strictly turbulent trajectory. Then there exists a continuous measure* m *supported on* $L(x_0)$ *which is ergodic and invariant with respect to* T.

Proof. Since $L(x_0)$ is compact and invariant, by the Kryloff-Bogoluboff theorem and by the Krein-Milman theorem there exists an ergodic invariant measure supported on $L(x_0)$. Lack of periodic points in $L(x_0)$ implies that the measure is continuous.

3. Existence of strictly turbulent trajectories.

THEOREM 1. *Let* T *be a continuous mapping from a topological Hausdorff space* X *into itself. Suppose that there exists a sequence* $A_0, \ldots, A_{N-1} (N > 1)$ *of compact, nonempty subsets of* X *such that*

$$\bigcap_{k=0}^{N-1} T(A_k) \supset \bigcup_{k=0}^{N-1} A_k \quad \text{and} \quad \bigcap_{k=0}^{N-1} A_k = \emptyset. \tag{1}$$

Then there exists a point $x_0 \in \bigcup_{k=0}^{N-1} A_k$ *such that the trajectory* $\mathcal{O}(x_0)$ *is turbulent.*

Remark. In the case $N = 2$ Theorem 1 was proved by A. Lasota and J. Yorke [3].

Proof of Theorem 1. Define a family $\{A_{k_0 \ldots k_n}\}$ of subsets of X by the formula

$$A_{k_0 \ldots k_n} = A_{k_0 \ldots k_{n-1}} \cap T^{-1}(A_{k_1 \ldots k_n}),$$

for $k_i = 0, 1, \ldots, N-1$, $i = 0, 1, \ldots, n$ and $n = 1, 2, 3, \ldots$ Using (1) it is easy to verify by the induction argument that the sets $A_{k_0 \ldots k_n}$ are nonempty, compact and that

$$T(A_{k_0 \ldots k_n}) = A_{k_1 \ldots k_n}. \tag{2}$$

The existence of a turbulent trajectory will be proved by induction.

1° $N = 2$. Write $A_0 = A$, $A_1 = B$.

Now choose an irrational number $\alpha \in (0, 1)$ and define a dyadic sequence $\{k_n\}$ by setting

$$k_n = \begin{cases} 0 & \text{if} \quad n\alpha \pmod 1 \in [0, \alpha) \\ 1 & \text{if} \quad n\alpha \pmod 1 \in [\alpha, 1). \end{cases}$$

$A_{k_0\dots k_n}$ is a decreasing sequence of nonempty compact sets. Thus the intersection

$$A_\infty = \bigcup_{n=0}^{\infty} A_{k_0\dots k_n}$$

is also a nonempty set. Choose $x_0 \in A_\infty$ and consider the trajectory $\mathcal{O}(x_0)$ and its limit set $L(x_0)$. From the definition of A_∞ it follows that

$$T^n(x_0) \in \begin{cases} A & \text{if} \quad n\alpha \,(\mathrm{mod}\ 1) \in [0, \alpha) \\ B & \text{if} \quad n\alpha \,(\mathrm{mod}\ 1) \in [\alpha, 1), \end{cases}$$

and consequently $\mathcal{O}(x_0) \subset A \cup B$. Since A and B are compact, this implies that $L(x_0) \subset A \cup B$. We shall show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A(T^k(x)) = \alpha \qquad \text{for } x \in L(x_0). \tag{3}$$

In fact, from the Weyl equipartition theorem it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{[0,\alpha)}(s + k\alpha \,(\mathrm{mod}\ 1)) = \alpha,$$

uniformly for all $s \in [0, 1)$. In particular,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{[0,\alpha)}((m+k)\alpha \,(\mathrm{mod}\ 1)) = \alpha$$

uniformly for all m. Consider a point $x \in \mathcal{O}(x_0)$, say $x = T^m(x_0)$. We have

$$1_A(T^k(x)) = 1_A(T^{m+k}(x_0)) = 1_{[0,\alpha)}((m+k)\alpha \,(\mathrm{mod}\ 1)),$$

which implies that the limit (3) exists uniformly for all $x \in \mathcal{O}(x_0)$. Since A and B are compact disjoint and the trajectory $\mathcal{O}(x_0)$ contained in $A \cup B$, the functions $1_A T^k(x)$ $k = 0, 1, \dots$ restricted to $\mathrm{cl}\,\mathcal{O}(x_0)$ are continuous. Thus the limit (3) exists also for $x \in \mathrm{cl}\,\mathcal{O}(x_0)$ and, in particular, for $x \in L(x_0)$. Now we are going to prove that the trajectory $\mathcal{O}(x_0)$ is turbulent. Since $L(x_0)$ is closed and $L(x_0) \subset A \cup B$, thus $L(x_0)$ is compact. To prove (ii) suppose that a point $\bar{x} \in L(x_0)$ is periodic. This implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A(T^k(\bar{x}))$$

is a rational number, which contradicts (3). The proof for $N = 2$ is completed.

2°. Assume the theorem true for a certain N. We shall prove that there exists a point $x_0 \in \bigcup_{k=0}^{N} A_k$ such that the trajectory $\mathcal{O}(x_0)$ is turbulent. From (1) and (2) it follows that

$$\bigcup_{k=0}^{N} A_k \subset T^{n+1}(A_{k_0\dots k_n}). \tag{4}$$

Now we shall consider two cases

(a) there exist n and sequences $\{k_0^i, \ldots, k_n^i\}$, $i = 0, 1, \ldots, N-1$, $k_j^i \in \{0, \ldots, N\}$ such that

$$\bigcap_{i=0}^{N-1} A_{k_0^i \ldots k_n^i} = \emptyset .$$

(b) for every n and sequences $\{k_0^i, \ldots, k_n^i\}$, $i = 0, 1, \ldots, N-1$, $k_j^i \in \{0, \ldots, N\}$ we have

$$\bigcap_{i=0}^{N-1} A_{k_0^i \ldots k_n^i} \neq \emptyset .$$

Assume (a). By (4) the subsets $A_{k_0^i \ldots k_n^i}$ and the mapping T^{n+1} satisfies (1). By the assumption of induction, there exists a turbulent trajectory for $S = T^{n+1}$. The set $\mathrm{cl}\,\mathcal{O}_S(x_0)$ is compact. This implies that $\mathrm{cl}\,\mathcal{O}_T(x_0)$ and $L_T(x_0)$ are compact. By Proposition 1 there exists a turbulent trajectory for T. Assume (b). Define the sequences

$$l_n^i = \begin{cases} i & \text{if} \quad n\alpha (\mathrm{mod}\ 1) \in [0, \alpha) \\ N & \text{if} \quad n\alpha (\mathrm{mod}\ 1) \in [\alpha, 1) . \end{cases}$$

Here α is the same as in 1°.

$\bigcap_{i=0}^{N-1} A_{l_0^i \ldots l_n^i}$ is a decreasing sequence of nonempty compact sets. Thus the intersections

$$A_\infty^N = \bigcap_{n=1}^{\infty} \bigcap_{i=0}^{N-1} A_{l_0^i \ldots l_n^i}$$

is also a nonempty set. Choose $x_0 \in A_\infty^N$. From the definition of A_∞^N it follows that

$$T^n(x_0) \in \bigcap_{i=0}^{N-1} A_{l_n^i} .$$

Write $A = \bigcap_{k=0}^{N-1} A_k$, $B = A_N$. Thus A and B are nonempty compact disjoint sets and

$$T^n(x_0) \in \begin{cases} A & \text{if} \quad n\alpha (\mathrm{mod}\ 1) \in [0, \alpha) \\ B & \text{if} \quad n\alpha (\mathrm{mod}\ 1) \in [\alpha, 1) . \end{cases}$$

Section 1° of the theorem implies the existence of the turbulent trajectory for T. The proof of the theorem is completed.

COROLLARY. *Let T be a continuous mapping from a topological Hausdorff space X into itself and suppose that there exists a sequence $A_0, \ldots, A_{N-1}$ $(N > 1)$ of subsets of X satisfying the assumptions of Theorem* 1. *Then there exists a continuous measure which is ergodic and invariant with respect to T.*

The proof follows immediately from Theorem 1 and Proposition 2.

Example 1. Let $\sigma^n = (p_0 \ldots p_n)$ be a closed geometrical non-degenerate n-simplex spanned on $p_0, \ldots, p_n$ $(n \geqslant 3)$. By $\partial_i \sigma^n$ we understand the $(n-1)$-face of σ^n which is opposite to the i-th vertex, that is p_i, and $\partial\sigma^n$ is the boundary of σ^n. Let $\bar{p} \in \partial\sigma^n$ and we consider some continuous mappings φ_i: $\partial_i\sigma^n \to \partial\sigma^n$ for $i = 0, ..., n$ satisfying the conditions

$$\varphi_i(\partial_i\sigma^n) = \partial\sigma^n \quad \text{and} \quad \varphi_i(\partial\partial_i\sigma^n) = \{\bar{p}\}.$$

We define φ: $\partial\sigma^n \to \partial\sigma^n$ in the following way:

$$\varphi(p) = \varphi_i(p) \quad \text{for} \quad p \in \partial_i\sigma^n.$$

Then φ is a continuous mapping from $\partial\varphi^n$ into itself. We put $A_i = \partial_i\sigma^n$ and $T = f\circ\varphi$, where f is an arbitrary continuous mapping from $\partial\sigma^n$ onto itself. Then T and $A_0, ..., A_n$ satisfy the assumptions of Theorem 1 ($N = n+1$). Hence there exists a strictly turbulent trajectory for T and, in consequence, there exists a continuous ergodic invariant measure for T.

4. Exactness. Let T be a continuous mapping from a topological Hausdorff space X into itself, and let m be a probabilistic measure invariant under T (defined on $\mathscr{B}(X)$- the σ-algebra of Borel subsets of X). The system $(X, \mathscr{B}(X), m; T)$ is called exact if for each Borel subset E the condition $E \in \bigcap_{n=1}^{\infty} T^{-n}(\mathscr{B}(X))$ implies $m(E)(1-m(E)) = 0$.

Example 2. Let $C = \prod_{i=0}^{\infty} \{0, 1\}_i$ be Cantor's set with the product topology of the discrete topologies on $\{0, 1\}$. Let μ be the product measure of the measures μ_i on $\{0, 1\}$ given by the formula $\mu_i(\{0\}) = \mu_i(\{1\}) = \frac{1}{2}$ and let S be the shift transformation on C given by the formula $S(\{x_i\}) = \{x_{i+1}\}$. Then the system $(C, \mathscr{B}(C), \mu; S)$ is exact.

We shall use the following simple observation.

LEMMA. *Let T be a continuous mapping from a topological Hausdorff space X into itself and there exists a sequence $(A_0, ..., A_{N-1})$ $(N > 1)$ of subsets of X satisfying the assumptions of Theorem* 1. *Then there exists a sequence $(A'_0, ..., A'_{N-1})$ of compact nonempty subsets of X such that*

$$T(A'_k) = \bigcup_{i=0}^{N-1} A'_i \quad \textit{and} \quad A'_k \subset A_k \tag{5}$$

for $k = 0, ..., N-1$.

Proof. Let $\mathscr{M}$ denote the family of all sequences $(A'_0, ..., A'_{N-1})$ of nonempty compact sets satisfying condition (1) and $A'_i \subset A_i$ for $i = 0, 1, ..., N-1$. The relation $(A'_0, ..., A'_{N-1}) \leqslant (B'_0, ..., B'_{N-1})$, if $A'_i \subseteq B'_i$ for $i = 0, ..., N-1$, partially orders this family. Zorn's lemma implies the existence of a minimal element of $\mathscr{M}$. It is easy to verify that this element satisfies condition (5). □

Now we are in position to prove the following

THEOREM 2. *Let T be a continuous mapping from a topological Hausdorff space X into itself. Suppose that there exist two nonempty compact disjoint sets A_0 and A_1 such that*

$$T(A_0) \cap T(A_1) \supset A_0 \cup A_1$$

and $\bigcap_{n=0}^{\infty} A_{k_0 \ldots k_n}$ contains exactly one point for every sequence $\{k_i\}$, where $k_i = 0, 1$. Then there exists a continuous measure m such that the system $(X, \mathscr{B}(X), m; T)$ is exact.

Proof. In virtue of Lemma we may assume that $T(A_0) = A_0 \cup A_1$ and $T(A_1) = A_0 \cup A_1$. This implies the following conditions

$$A_{k_0 \ldots k_n 0} \cup A_{k_0 \ldots k_n 1} = A_{k_0 \ldots k_n} \tag{6}$$

$$A_{k_0 \ldots k_n} \cap A_{k'_0 \ldots k'_n} = \emptyset \quad \text{for} \quad (k_0 \ldots k_n) \neq (k'_0 \ldots k'_n). \tag{7}$$

We define a function $\varphi\colon A_0 \cup A_1 \to C$, where C is Cantor's set defined in Example 2, by the formula

$$\varphi(x) = \{k_i\} \quad \text{if} \quad x \in \bigcap_{n=0}^{\infty} A_{k_0 \ldots k_n}$$

By (6) and (7) the definition of φ is correct. For every sequence $\{k_i\}$, the set $\bigcap_{n=0}^{\infty} A_{k_0 \ldots k_n}$ contains exactly one point, whence φ is a homeomorphism. We observe that

$$\varphi \circ T = S \circ \varphi. \tag{8}$$

Now we define measure $\overline{m}$ by the formula

$$\overline{m}(E) = \mu(\varphi(E)) \quad (E \text{ Borel subset of } A_0 \cup A_1). \tag{9}$$

From (8) and (9) it follows that the system $(A_0 \cup A_1, \mathscr{B}(A_0 \cup A_1), \overline{m}; T)$ is isomorphic to $(C, \mathscr{B}(C), \mu; S)$. Thus $(A_0 \cup A_1, \mathscr{B}(A_0 \cup A_1), \overline{m}; T)$ is exact. The extension m of $\overline{m}$ from $\mathscr{B}(A_0 \cup A_1)$ onto $\mathscr{B}(X)$ is the desired measure. □

Remark. The theorem is also true if we assume that except for a countable number of sequences $\{k_i\}$ the set $\bigcap_{n=0}^{\infty} A_{k_0 \ldots k_n}$ contains exactly one point. The theorem in this form can be applied to pseudo-r-adic transformations (see Lasota [4]).

Example 3. Let X be the space of all continuous real valued functions $x\colon [0, 1] \to R$ with the supremum norm topology, and let $T\colon X \to X$ be the transformation given by the formula

$$T(x)(t) = rx\left(\frac{t}{r}\right) \quad 0 \leqslant t \leqslant 1,$$

where $r > 1$. From Theorem 2 it follows that for each $r > 1$ there exists a continuous measure m such that the system $(X, \mathscr{B}(X), m; T)$ is exact. In fact, denote by X_1 the subspace of X which contains all functions satisfying

$$x(0)=0 \quad |x(t)-x(s)| \leqslant |t-s| \quad 0 \leqslant s, t \leqslant 1,$$

and write

$$A_0 = \left\{ x \in X_1 \colon x(t) = x\left(\frac{1}{r}\right), \qquad \frac{1}{r} \leqslant t \leqslant 1 \right\}$$

$$A_1 = \left\{ x \in X_1 \colon x(t) = x\left(\frac{1}{r}\right) + t - \frac{1}{r}, \quad \frac{1}{r} \leqslant t \leqslant 1 \right\}.$$

It is easy to see that

$$T(A_0) = T(A_1) = X_1 \supset A_0 \cup A_1 .$$

By the induction argument we get

$$A_{k_0 \dots k_n} = \left\{ x \in X_1 \colon x'(t) = k_i, \quad t \in \left(\frac{1}{r^{i+1}}, \frac{1}{r^i}\right), \quad i = 0, 1, \dots, n \right\}.$$

For $x, y \in A_{k_0 \dots k_n}$ we have

$$\|x-y\| = \sup_{t \in [0,1]} |x(t) - (t)| = \sup_{t \in [0, 1/r^{n+1}]} |x(t) - y(t)| \leqslant \frac{1}{r^{n+1}} .$$

This implies that $\bigcap_{n=0}^{\infty} A_{k_0 \dots k_n}$ contains exactly one point. Since A and B are nonempty and compact and $A \cap B = \emptyset$ all the assumptions of Theorem 2 are satisfied.

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