UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXVI
1987

On Some Solution of the Diffusion Equation

by Eugeniusz BOBULA

Let us consider the diffusion problem. The process is described by a parabolic equation in the space of distributions [1]:

$$\frac{\partial \mathbf{P}}{\partial t}-\frac{\partial^2 \mathbf{P}}{\partial x^2}=2\frac{\partial p}{\partial x}\bigg|_{x=0}\cdot\delta_0 \tag{1}$$

$$D\ni\varphi\to T_p=\int_{-\infty}^{\infty}p(x,t)\varphi(x)dx\equiv \mathbf{P}$$

Eq. (1) is a particular form of Eq. (12), which was considered in work [1], for $c(x,t)=0$. Let us find the solution of Eq. (1) for following conditions

$$p(x,0)=\Psi(x),\quad \Psi(x)=\Psi(-x)\geqslant 0,\quad \Psi\in C^0 \tag{2}$$

$$p(\pm\lambda(t),t)=0,\quad \lambda(t)=\sqrt{r-t},\quad r=\text{const}>0 \tag{3}$$

$$\int_{-\lambda(t)}^{\lambda(t)}p(x,t)dx=\text{const} \tag{4}$$

The solution vanishing on the parabola has to be sought in the form

$$p(x,t)=\frac{1}{\sqrt{r-t}}\sum_{j=0}^{\infty}a_j\left(1-\frac{|x|}{\sqrt{r-t}}\right)^{j+1} \tag{5}$$

After inserting (5) into (1) e.g. for $x>0$ we get a second order difference equation for the coefficients a_j of the series. We have thus

$$a_{j+2}+\frac{1}{2(j+3)}a_{j+1}-\frac{1}{2(j+3)}a_j=0 \tag{6}$$

Owing to the symmetry of the solution we may assume $0<x<\sqrt{r-t}$. According to the Poincaré theorem about the equation

$$f(j+2)+P_1(j)f(j+1)+P_0(j)f(j)=0$$

if there exist $\lim_{j\to\infty} P_1(j) = \alpha_1$ and $\lim_{j\to\infty} P_0(j) = \alpha_0$, then $\lim_{j\to\infty} \frac{f(j+1)}{f(j)}$ is one of the roots of the equation $q^2+\alpha_1 q+\alpha_0 = 0$. In our case $\alpha_1 = 0$, $\alpha_0 = 0$. Consequently, $q = 0$. Thus $\lim_{j\to\infty} \frac{f(j+1)}{f(j)} = 0$. The last expression is the reciprocal of the convergence radius of the series (5). Therefore, this series is uniformely convergent for $0 < 1 - \frac{x}{\sqrt{r-t}} < 1$.

Moreover

$$\int_0^{\sqrt{r-t}} \frac{1}{\sqrt{r-t}} \sum_{j=0}^{\infty} a_j \left(1 - \frac{x}{\sqrt{r-t}}\right)^{j+1} dx = \sum_{j=0}^{\infty} \int_0^{\sqrt{r-t}} \left(1 - \frac{x}{\sqrt{r-t}}\right)^{j+1} \frac{dx}{\sqrt{r-t}}.$$

After the change of variables $y = \frac{x}{\sqrt{r-t}}$ we get, instead of the integral form, the series without any term depending on t. Thus $\int_0^{\sqrt{r-t}} p(x, t)dx$ does not vary with time.

$$p(x, 0) = \frac{1}{\sqrt{r}} \sum_{j=1}^{\infty} a_j \left(1 - \frac{|x|}{\sqrt{r}}\right)^{j+1} \text{ evidently. } \frac{d\lambda(t)}{dt} = \frac{d\sqrt{r-t}}{dt} < 0.$$

References

[1] E. Bobula, *Derivation of the diffusion equation in the space of distributions*, Zesz. Nauk. Uniw. Jagiell. Prace Mat. Z. 22, 1981.

Received March 20, 1982.