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Some Corollaries of Gromov's Theorems

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Preface. This paper is based on Gromov's thesis published in 1969 [1]. Since it did not contain the exact proofs of some of the results we present some of them in this paper. Gromov's theorems generalize a number of other results obtained by Hirsch, Philips and Whitney.

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Chapter I. In this chapter we shall construct a smooth fibre bundle and formulate the first Gromov's theorem. Let us consider smooth (i.e. C^∞ differentiable) manifolds M, F, $\dim M = n$, $\dim F = k$. Let the group

$$L_n^1 = \{j_0^1 f \mid f\colon (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0) \text{ is a local diff.}\}$$

acts smoothly on the left on F

$$\mu\colon L_n^1 \times F \to F$$

Let $FM(M, L_n^1)$ be the principal fibre bundle of the linear frames of M and E be the associated bundle with fibre F. Let $J^r E := \bigcup_{x \in M} \{j_x^r f \mid f\colon U_x \to E \text{ is a smooth local section}\}$. So, we have the canonical projection π

$$\pi\colon J^r E \to M.$$

PROPOSITION 1.1 *There exists the canonical differential structure of a fibre bundle on $J^r E \xrightarrow{\pi} M$ with structure group L_n^{r+1} and standard fibre $J^r F$, where*

$$J^r F := \{j_0^r f \mid f\colon (\boldsymbol{R}^n, 0) \to F \text{ is a smooth mapping}\}.$$

Proof. If

$$(U, \varphi), (V, \Psi) \in \operatorname{Atl}(M)$$

then we have the mappings $\tilde{\varphi}, \tilde{\psi}$ such that

$$\tilde{\varphi}(x, j_0^r f) = j_x^r \tilde{f}, \quad \tilde{\Psi}(x, j_0^r f) = j_x^r \tilde{\tilde{f}}$$

where

$$\tilde{f}(y) := \langle (\partial_1, \ldots, \partial_n)_y, f(\varphi(y) - \varphi(x)) \rangle$$

and

$$\tilde{\tilde{f}}(y) := \langle(\bar{\partial}_1, \ldots, \bar{\partial}_n)_y, f(\psi(y)-\varphi(x))\rangle$$

and $(\partial_1, \ldots, \partial_n)$ is the canonical frame induced by φ, $(\bar{\partial}_1, \ldots, \bar{\partial}_n)$ is the canonical frame induced by ψ. We have

$$(\partial_1, \ldots, \partial_n)_y = (\bar{\partial}_1, \ldots, \bar{\partial}_n)_y \cdot \left(\frac{\partial \psi^j \circ \varphi^{-1}}{\partial u^i}(\varphi(y))\right)_{i,j=1,\ldots,n}.$$

So, we obtain

$$\langle(\partial_1, \ldots, \partial_n)_y, f(\varphi(y)-\varphi(x))\rangle = \left\langle(\bar{\partial}_1, \ldots, \bar{\partial}_n)\cdot\left(\frac{\partial \psi^j \circ \varphi^{-1}}{\partial u^i}(\varphi(y))\right)_{i,j=1,\ldots,n}, f(\varphi(y)-\varphi(x))\right\rangle.$$

Hence

$$\tilde{\psi}^{-1} \circ \tilde{\varphi}(x, j_0^r f) = (x, j_0^r \hat{F})$$

where $T_a x = a+x$ and

$$\hat{F}(w) = \mu\big(j_0^1[T_{-(w+\psi(x))} \circ \psi \circ \varphi^{-1} \circ T_{\varphi(\psi^{-1}(w+\psi(x)))}], f \circ T_{-\varphi(x)} \circ \varphi \circ \psi^{-1} \circ T_{\psi(x)}(w)\big)$$

Thus it is natural to define an action of L_n^{r+1} on the manifold $J^r F$

$$\tilde{\mu}\colon L_n^{r+1} \times J^r F \to J^r F$$

in the following way

$$\tilde{\mu}(j_0^{r+1}\alpha, j_0^r f) := j_0^r h$$

where

$$h(w) = \mu\big(j_0^1(T_{-w} \circ \alpha \circ T_{\alpha^{-1}(w)}), f(\alpha^{-1}(w))\big)$$

Using standard computation we check that

1. $\tilde{\mu}$ is well defined,
2. $\tilde{\mu}$ is an action of L_n^{r+1} on $J^r F$ on the left,
3. $\tilde{\mu}$ is smooth.

On the other hand we define a family of functions $\varrho_{\psi\varphi}$ in the following way:

$$\varrho_{\psi\varphi}\colon U \cap V \to L_n^{r+1}$$

$$\varrho_{\psi\varphi}(x) := j_0^{r+1}(T_{-\psi(x)}\,;\, \psi \circ \varphi^{-1} \circ T_{\varphi(x)}).$$

It is clear that

1. $\varrho_{\psi\varphi}$ are smooth functions,
2. $\varrho_{\varphi\varphi} \equiv 1 \in L_n^{r+1}$,
3. if $(U, \varphi), (V, \psi), (W, \chi) \in \mathrm{Atl}(M)$, $U \cap V \cap W \neq \emptyset$ then for every $x \in U \cap V \cap W$

$$\varrho_{\chi\psi}(x) \cdot \varrho_{\psi\varphi}(x) = \varrho_{\chi\varphi}(x)$$

Moreover the following equality holds

$$\tilde{\psi}^{-1} \circ \tilde{\varphi}(x; j_0^r f) = (x, \varrho_{\psi\varphi}(x) \cdot j_0^r f)$$

Therefore we obtain that the family

$$\{(U \times J^r F, \tilde{\varphi})\}_{(U,\varphi) \in \mathrm{Atl}(M)}$$

defines the differential structure of the fibre bundle J^rE with standard fibre J^rF and with structure group L_n^{r+1}.

COROLLARY 1.2 *If $H \subset J^rF$ is an open subset and $\tilde{\mu}(L_n^{r+1}, H) \subset H$, then it induces an open subbundle of the bundle J^rE with standard fibre H.*

We shall denote this bundle

$$\pi\colon J^rE_H \to M$$

Notation: if $p\colon \eta \to N$ is a fibre bundle, then $\Gamma^k(\eta)$ will denote the set of all C^k-sections of this bundle.

As it was mentioned,

$$\pi\colon E \to M$$

is a fibre bundle associated with F. The set $\Gamma^\infty(E)$ is a topological space with the weak C^k-topology. Let d^k be the mapping

$$d^k\colon \Gamma^\infty(E) \to \Gamma^0(J^rE)$$

defined by

$$d^k(\sigma)(x) = j_x^k\sigma \quad \text{for } \sigma \in \Gamma^\infty(E)$$

Let $H \subset J^rF$ be an open set invariant under the action of L_n^{r+1}, then we shall use the following notation

$$B_H := (d^k)^{-1}[\Gamma^0(J^rE_H)]$$

The main result concerning the map d^k for an open manifold is the following theorem

THEOREM 1.3 *If M is an open manifold (that is if M has no compact components) and if $H \subset J^kF$ is an open L_n^{k+1}-invariant subset, then the map*

$$d^k\colon B_H \to \Gamma^0(J^kE_H)$$

is a weak homotopy equivalence.

For the proof see [1].

Chapter II. In this chapter w shall present two corollaries of Gromov's theorem (theorem 2.2 and 2.4). Lemma 2. will be used to prove Theorem 2.4. This theorem concerns transversal mappings on the open manifold. The second theorem gives a sufficient condition for a distribution on an open manifold to be homotopic to a foliation.

Let M, N be smooth manifolds and let η be a subbundle of the bundle TN. Let Φ be a smooth function

$$\Phi\colon M \to N$$

DEFINITION 2.1 The function Φ is *transversal* to η (we shall write $\Phi \pitchfork \eta$) iff for every $x \in M$

$$\operatorname{im} d_x\Phi + \eta_{\Phi(x)} = T_{\Phi(x)}N$$

Notation:

$$C_\eta^\infty(M, N) = \{f \in C^\infty(M, N)\colon f \pitchfork \eta\}$$

$$\mathrm{Hom}_\eta(TM, TN) = \{F \in \mathrm{Hom}(TM, TN)\colon \forall_{x \in M} \mathrm{im} F_x + \eta_{\bar{F}(x)} = T_{\bar{F}(x)} N\}$$

The set

$$C_\eta^\infty(M, N)$$

is equiped with the weak C^1-topology and the set

$$H_\eta(TM, TN)$$

is equiped with the compact-open topology. There is the natural continuous mapping

$$d\colon C_\eta^\infty(M, N) \to \mathrm{Hom}_\eta(TM, TN)$$

$$(df)(x) = d_x f$$

THEOREM 2.2 *If M is an open manifold, then the mapping d is a weak homotopy equivalance.*

Proof. Let

$$H = \{j_0^1 f \mid f\colon \boldsymbol{R}^n \to N \text{ and } \mathrm{im} d_0 f + \eta_{f(x)} = T_{f(x)} N\};$$

then $H \subset J^1 N$ and H is an open subset. Let μ be the trivial action

$$\mu\colon L_n^1 \times N \to N$$

so the induced action $\tilde{\mu}$ acts as follows

$$\tilde{\mu}(j_0^2 \alpha, j_0^1 f) = j_0^1(f \circ \alpha^{-1})$$

Hence we obtain that H is L_n^2-invariant, thus by the Gromov's theorem the function

$$d^1\colon B_H \to \Gamma^0(J^1 E_H)$$

is a weak homotopy equivalence. The bundle E is a trivial bundle since μ acts trivially, hence

$$B_H \subset C^\infty(M, N)$$

Thus, we have a function

$$A_1\colon B_H \to C_\eta^\infty(M, N)$$

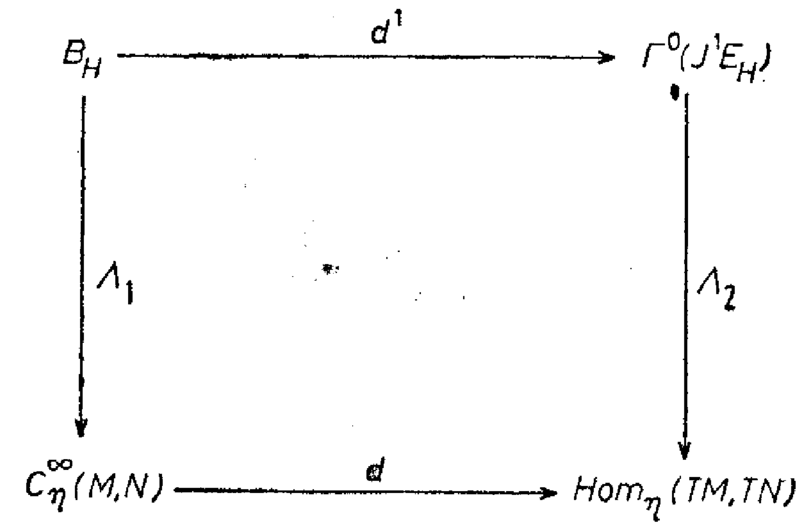
From the definition of topology in B_H it follows that A_1 is a homeomorphism. Now we shall define a function

$$A_2\colon \Gamma^0(J^1 E_H) \to \mathrm{Hom}_\eta(TM, TN)$$

$$(A_2 \sigma)(x) = d_x f, \quad \text{where } \sigma(x) = j_x^1 f.$$

It easy to see that A_2 is a homeomorphism.

Finally, since the following diagram commutes



d is a weak homotopy equivalence.

LEMMA 2.3 *If*

$$p\colon \zeta \to M$$

is a smooth vector bundle which has discrete structural group G then there exists an involutive subbundle

$$\tilde{\zeta} \subset T\zeta$$

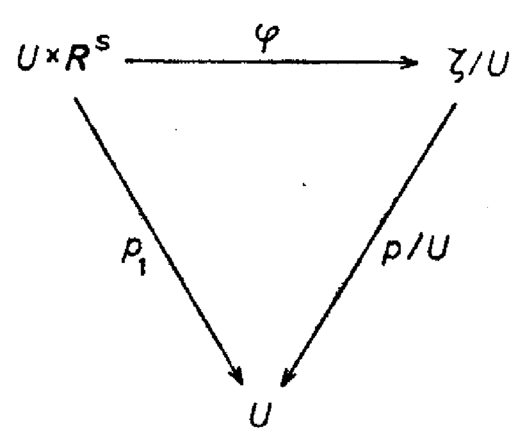
such that for every $a \in \zeta$

$$\tilde{\zeta}_a \oplus T_a(p^{-1}(p(a)) = T_a\zeta$$

Proof. Let (U, φ) be a local trivialization of the bundle

$$p\colon \zeta \to M,$$

so the following diagram



commutes where s is the dimension of the fibre. Let $a = \varphi(x, \lambda)$, then we shall define

$$\tilde{\zeta}_a := (d_{(x,\lambda)}\varphi)(T_xM)$$

We have to show that

1. $\tilde{\zeta}$ is well defined,
2. $\tilde{\zeta}$ is a smooth involutive subbundle of $T\zeta$,
3. for every $a \in \zeta$

$$\tilde{\zeta}_a \oplus T_a p^{-1}(p(a)) = T_a\zeta$$

The points 2 and 3 are obvious. Let $a = \psi(x, \lambda)$ and let (V, ψ) be another trivialization of the bundle

$$p\colon \zeta \to M.$$

So we have

$$(d_{(x,\lambda)}\varphi)[\tau \to (f(\tau), \lambda)] = [\tau \to \varphi(f(\tau), \lambda)],$$

$$\begin{aligned}(d_a\psi^{-1})[\tau \to \varphi(f(\tau), \lambda)] &= [\tau \to (\psi^{-1}\circ\varphi)(f(\tau), \lambda)] \\ &= [\tau \to (f(\tau), \Phi_{\psi\varphi}(f(\tau))\cdot\lambda)] = [\tau \to (f(\tau), \Phi_{\psi\varphi}(x)\lambda)],\end{aligned}$$

where $\Phi_{\psi\varphi}$ is the transition mapping. The last equality holds because the mapping

$$\Phi_{\psi\varphi}\colon U \cap V \to G$$

is locally constant. Thus we have shown that

$$d_{(x,\lambda)}(\psi^{-1}\circ\varphi)(T_x M) \subset T_x M$$

so 1. is true. The following theorem concerns foliations (see Definition 3.1)

THEOREM 2.4. *A distribution ξ on an open manifold M whose normal bundle TM/ξ admits a reduction to a discrete group is homotopic to a foliation.*

Proof. We shall define a morphism of vector bundles

$$F\colon TM \to T(TM)$$

by

$$F(Z) = [\tau \to Z\cdot\tau]$$

If $Z \in T_x M$, then

$$F(Z) \in T_{o_x}(TM).$$

It is easy to see that F is a smooth monomorphism of vector bundles and for every $x \in M$

$$\operatorname{im} F_x = T_{o_x}(T_x M).$$

Let $\zeta = TM/\xi$; we know that the fibre bundle

$$p\colon \zeta \to M$$

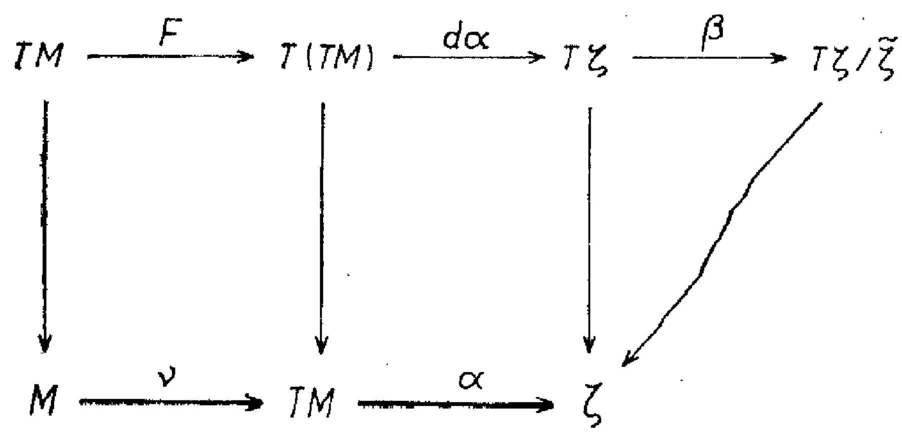
has a discrete structure group, then by lemma 2.3 there exists an involutive subbundle $\tilde{\zeta} \subset T\zeta$ such that for every $a \in \zeta$

$$\tilde{\zeta}_a \oplus T_a p^{-1}(p(a)) = T_a\zeta.$$

We shall use the following canonical projections α, β

$$\alpha\colon TM \to \zeta \quad \text{and} \quad \beta\colon T\zeta \to T\zeta/\tilde{\zeta}.$$

We have the following commutative diagram



Here ν is the zero vectorfield. We shall show that

1. for every $x \in M$ the mapping $(\beta \circ d\alpha \circ F)_x$ is surjective on fibres
2. $\ker(\beta \circ d\alpha \circ F) = \xi$.

At first we notice an obvious fact that if $\eta_1 \subset \eta_2$ are a vector bundle and its subbundle, then the canonical projection

$$\eta_2 \to \eta_2/\eta_1$$

is a submersion. Thus

$$d\alpha\colon T_{o_x}(TM) \to T_{o_x}\zeta$$

is a surjection, moreover

$$(d_{o_x}\alpha)[T_{o_x}(T_x M)] \subset T_{o_x}\zeta_x$$

since

$$d\alpha\colon T_{o_x}(T_x M) \to T_{o_x}\zeta_{o_x}$$

is a surjection. So we have proved the first point. On the other hand we have that

$$T_{o_x}(\xi_x) \subset \ker(d_{o_x}\alpha)\,.$$

It implies that

$$\forall x \in M \;\; \xi_x \subset \ker(\beta \circ d\alpha \circ F)_x\,.$$

The opposite inclusion is true because of the surjectivity of the function

$$(\beta \circ d\alpha \circ F)_x$$

From Theorem 2.2 we obtain that there exists a continuous function

$$\Delta\colon [0, 1] \to \mathrm{Hom}_{\xi}(TM, T\zeta)$$

and there exists a smooth mapping

$$f\colon M \to \zeta$$

such that $f \in C^{\infty}_{\xi}(M, \zeta)$, $\Delta(0) = df$ and $\Delta(1) = d\alpha \circ F$.

Let $G_m(M)$ be the Grassmann fibre bundle, where $m = \dim \xi_x$. We shall define the following mapping A

$$A\colon \mathrm{Hom}_{\xi}(TM, T\zeta) \to \Gamma^0(G_m(M))$$

If

$$H \in \mathrm{Hom}_{\xi}(TM, T\zeta)$$

then $A(H)$ is a section of the Grassmann fibre bundle such that

$$A(H)(x) = H_x^{-1}(\tilde{\zeta})$$

The continuity of the function A can be shown by using standard but rather complicated calculus.

Finally, we obtain the following mapping

$$A \circ \Delta\colon [0, 1] \to \Gamma^0(G_m(M))$$

where

$$A \circ \Delta(1) = (d\alpha \circ F)^{-1}(\tilde{\zeta}) = \ker(\beta \circ d\alpha \circ F) = \xi$$

and

$$(A \circ \Delta)(0) = (df)^{-1}(\tilde{\zeta}).$$

The distribution

$$(df)^{-1}(\tilde{\zeta})$$

is involutive so ξ is homotopic to a foliation.

Chapter III. In his work [1] Gromov gives another variant of Theorem 1.2. In this chapter we shall be concerned with a corollary of the second variant. At first we shall recall the definition of a foliation and then construct an associated fibre bundle; together with the definition of an injective set it will allowed us to state the second Gromov's theorem. Then we shall give an example of an injective set and this will be applied in corollary 3.11.

Let M, F be smooth manifolds where $\dim M = n+k$

DEFINITION 3.1. *A smooth n-dimensional foliation on M* is a pair (M, S), where S fulfils the following conditions

1. $S \subset \mathrm{Atl}(M)$,
2. if $(U, \psi), (V, \varphi) \in S$, then for every $z \in U \cap V$

$$U \cap \{w \in V | \varphi^{n+1}(w) = \varphi^{n+1}(z), \ldots, \varphi^{n+k}(w) = \varphi^{n+k}(z)\}$$
$$= V \cap \{w \in U | \psi^{n+1}(w) = \psi^{n+1}(z), \ldots, \psi^{n+k}(w) = \psi^{n+k}(z)\},$$

3. the set S is maximal with respect to the properties 1, 2.

Let $C^\infty(M, S, \boldsymbol{R})$ be the set of mappings f from M to $\boldsymbol{R}$ such that for every

$$\gamma \in N^n \times \{0\}^k$$

and for every $(U, \varphi) \in S$ the mapping

$$D^\gamma(f \circ \varphi^{-1})$$

is well defined and continuous. We define the set $C^\infty(M, S, F)$ analogously. Now we shall define a bundle

$$[S, F]^1 := \bigcup_{x \in M} \{j^1_x f \mid f \in C^\infty(U_x, S, F)\}$$

There is the natural projection π

$$\pi\colon [S, F]^1 \to M \text{ such that } \pi(j^1_x f) = x.$$

Let (U, φ) be an arbitrary chart from S, then it induces the mapping

$$\tilde{\varphi}\colon U \times J^1 F \to \pi^{-1}(U)$$

such that

$$\tilde{\varphi}(x, j^1_0 f) = j^1_x (f \circ \mathrm{pr} \circ T_{-\varphi(x)} \circ \varphi)$$

where

$$\mathrm{pr}\colon \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n,\ \mathrm{pr}(x, y) = x \quad \text{and} \quad T_w\colon \mathbf{R}^{n+k} \to \mathbf{R}^{n+k},\ T_w(z) = z + w.$$

PROPOSITION 3.2. *The family*

$$\{(U \times J^1 F, \tilde{\varphi}) \mid (U, \varphi) \in S\}$$

defines the structure of a continous fibre bundle on $[S, F]^1$ *with standard fibre* $J^1 F$ *and with structure group* L^1_n.

Proof. Let $(U, \varphi), (V, \psi) \in S$. These charts induce the mapping

$$\tilde{\psi}^{-1} \circ \tilde{\varphi}$$

which acts as follows

$$\tilde{\psi}^{-1} \circ \tilde{\varphi}\colon (U \cap V) \times J^1 F \to (U \cap V) \times J^1 F$$

So we have the following equality

$$\tilde{\psi}^{-1} \circ \tilde{\varphi}(x, j^1_0 f) = j^1_0 (f \circ \mathrm{pr} \circ T_{-\varphi(x)} \circ \varphi \circ \psi^{-1} \circ T_{\psi(x)} \circ u)$$

where

$$u\colon \mathbf{R}^n \hookrightarrow \mathbf{R}^{n+k}.$$

On the other hand we have the natural action of L^1_n on $J^1 F$

$$\vartheta\colon L^1_n \times J^1 F \to J^1 F$$

$$\vartheta(j^1_0 g, j^1_0 f) = j^1_0 (f \circ g^{-1}).$$

Finally for every $(U, \varphi), (V, \psi) \in S$ let $\varrho_{\psi\varphi}$ be the following mapping

$$\varrho_{\psi\varphi}\colon U \cap V \to L^1_n$$

$$\varrho_{\psi\varphi}(x) := j^1_0 (\mathrm{pr} \circ T_{-\psi(x)} \circ \psi \circ \varphi^{-1} \circ T_{\varphi(x)} \circ u)$$

The mappings have the following properties

1. $\varrho_{\psi\varphi}$ are continuous,
2. $\tilde{\psi}^{-1} \circ \tilde{\varphi}(x, \lambda) = (x, \varrho_{\psi\varphi}(x) \cdot \lambda)$,

3. $\varrho_{\psi\psi} \equiv 1 \in L_n^1$.

4. $\varrho_{\chi\psi}(x) \cdot \varrho_{\psi\varphi}(x) = \varrho_{\chi\varphi}(x)$.

Hence we obtain a structure of a continuous fibre bundle on $[S, F]^1$ with standard fibre J^1F.

Let now w be the canonical inclusion

$$w\colon \mathbf{R}^n \hookrightarrow \mathbf{R}^{n+1}$$

and let $J_*^1 F$ be a set of 1-jets of smooth functions at the origin of $n+1$-dimensional Euclidian space to the manifold F. There is the natural function

$$q\colon J_*^1 F \to J^1 F,$$
$$q(j_0^1 f) := j_0^1(f \circ w).$$

DEFINITION 3.3. If H is any subset of J^1F, then by $\Sigma(H)$ we shall denote a subset of $J_*^1 F$ such that the following conditions holds:

1. $\Sigma(H) \subset q^{-1}(H)$,
2. $L_{n+1}^1 \cdot \Sigma(H) \subset \Sigma(H)$,
3. $\Sigma(H)$ is *maximal* with respect to the properties 1,2.

COROLLARY 3.4. *For every $H \subset J^1F$ there exists the unique set $\Sigma(H)$.*

DEFINITION 3.5. The set $H \subset J^1F$ is *injective* iff the following three conditions hold:

1. the projection $q\colon \Sigma(H) \twoheadrightarrow H$ is a surjection,
2. the set H is open in J^1F and L_n^1-invariant,
3. the projection $q\colon \Sigma(H) \to H$ is a Serre fibration.

Let H be an open, L_n^1-invariant subset of J^1F so by Proposition 3.2 H induces a subbundle of the bundle $[S, F]^1$; it will be denoted $[S, F, H]^1$. We have the natural mapping

$$\partial_M\colon C^\infty(M, S, F) \to \Gamma^0([S, F]^1),$$

where

$$\partial_M(f) = \sigma \quad \text{and} \quad \sigma(x) = j_x^1 f$$

Let us use the following notation

$$B(S, F, H) := \partial_M^{-1}[\Gamma^0([S, F, H]^1)].$$

We are thus in a position to formulate the main theorem of this paragraph, which was proved by Gromov in his thesis.

THEOREM 3.6. *If the set H is injective, then the mapping*

$$\partial_M\colon B(S, F, H) \to \Gamma^0([S, F, H]^1)$$

is a weak homotopy equivalence.

For proof see [1].

Example 3.7. Let $\xi \subset TF$ be a subbundle of the bundle TF. Let P be the canonical projection

$$P\colon TF \to TF/\xi.$$

Let us denote H_0 a subset of J^1F such that

$$H_0 := \{j_0^1 f \in J^1F | P \circ d_0 f \text{ is an injection}\}$$

We shall show that if the dimension of a fibre in ξ is r, $\dim F = m$, and the following inequality holds

$$n < m - r$$

then the set H_0 is injective. It has to be shown that:

1. the projection

$$q \colon \Sigma(H_0) \to H_0$$

is a surjection
2. the set H_0 is open in J^1F and L^1_{n+1}-invariant,
3. the projection

$$q \colon \Sigma(H_0) \to H_0$$

is a Serre fibration.

The point 2 is obvious. Let us define a subset $\tilde{\Sigma}(H_0)$ of $J^1_* F$

$$\tilde{\Sigma}(H_0) := \{j_0^1 f \in J^1_* F | P \circ d_0 f \text{ is an injection}\}$$

It will be shown that

$$\tilde{\Sigma}(H_0) = \Sigma(H_0)$$

Obviously

$$\tilde{\Sigma}(H_0) \subset q^{-1}(H_0)$$

and $\tilde{\Sigma}(H_0)$ is L^1_{n+1}-invariant. Hence

$$\tilde{\Sigma}(H_0) \subset \Sigma(H_0).$$

Suppose that

$$j_0^1 f \in \Sigma(H_0) \setminus \tilde{\Sigma}(H_0),$$

this means that $P \circ d_0(f \circ w)$ is an injection and $P \circ d_0 f$ is not. So there exists a diffeomorphism

$$\alpha \colon (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}^{n+1}, 0)$$

such that $P \circ d_0(f \circ \alpha \circ w)$ is not an injection, which is impossible. So we obtain that

$$\tilde{\Sigma}(H_0) = \Sigma(H_0).$$

Since

$$\dim F - \dim \xi > \dim M,$$

we get that the projection

$$q \colon \Sigma(H_0) \to H_0$$

is a surjection. To prove that q is a Serre fibration the following notation will be used: if V is a subspace of $\mathbf{R}^m$ then

$$\mathrm{gl}(n, m;\, n, V) := \{A \in \mathrm{gl}(n, m) | \ker A = 0 \text{ and } \operatorname{im} A \cap V = 0\}.$$