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On Boundary Value Problems for Differential Equations with a Retarded Argument

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1. Introduction. The boundary value problem

$$\dot{x} = f(t, x_t)$$
$$Ax = \alpha$$

of a functional differential equation with a retarded argument was considered in [5] for a function f: $[a, b] \times C^0 \ni (t, \varphi) \to f(t, \varphi) \in R^d$ fulfilling the Carathéodory conditions and for a linear continuous operator A defined on the Banach space W of absolutely continuous functions with values in R^d. It was stated there that if for every operator A belonging to an open set $\Omega \subset \mathscr{L}$ ($\mathscr{L}$ denotes the space of linear continuous operators from W to R^d) and for every vector $\alpha \in R^d$ the above problem has at most one solution, then it has exactly one solution for every $A \in \Omega$ and every $\alpha \in R^d$. Moreover, the existence and uniqueness theorem was proved for the operator A_0 which represents the Cauchy problem.

In this paper the above results and the continuous dependence theorem contained in [5] are transformed to the case in which the space C^0 is replaced by the space of Lipschitzean functions.

Since the boundary value problems given by difference equations may be replaced by the boundary value problems for differential equations with a delayed argument we obtain as a corollary of our theorems the existence and uniqueness theorem of the solutions of the boundary value problems for difference equations. As a corollary we obtain also the theorem concerning the convergence of solutions of boundary value problems of difference equations to the solution of the boundary value problem for a given differential equation approximated by these difference equations.

One theorem dealing with the convergence of the solutions of the boundary value problems for differential equations with a retarded argument to the solution of a given ordinary differential equation was also proved by J. Myjak in [4]. However, that theorem is not of the same type. The author proved in it that under some assumptions the solution of the boundary value problem for the differential equation with a retarded argument and with fixed right-hand side converges to the solution of the ordinary differential equation with the same right-hand side with delays tending to zero. Our theorem includes the case where right-hand sides of the equations are changing and are convergent.

2. Definitions and notations. Let $W = W([a, b], R^d)$ be the space of absolutely continuous functions $x\colon [a, b] \to R^d$ endowed with the norm

$$||x||_w = |x(a)| + \int_a^b |\dot{x}(t)|\,dt\,.$$

The space of linear continuous operators from W to R^d with the norm topology is denoted by $\mathscr{L} = \mathscr{L}(W, R^d)$.

Let h_0 be a positive number. We shall denote by $C^0 = C^0([a-h_0, a], R^d)$ the space of all continuous functions from $[a-h_0, a]$ into R^d with the norm

$$||x||_{C^0} = \sup\{|x(t)|\colon\ t \in [a-h_0, a]\}$$

and by $L = L([a-h_0, a], R^d)$ the space of Lipschitz functions from $[a-h_0, a]$ into R^d, with the norm

$$||x||_L = \sup\{|x(t)|\colon\ t \in [a-h_0, a]\} + \sup\left\{\frac{|x(t')-x(t)|}{|t'-t|}\colon\ a-h_0 \leqslant t < t \leqslant a\right\}.$$

We remind that the function $f\colon [a, b]\times L \ni (t, \varphi) \to f(t, \varphi) \in R^d$ fulfils *the Carathéodory conditions* if

(i) it is measurable with respect to t for every fixed φ,

(ii) it is continuous with respect to φ for every fixed t,

(iii) for every bounded set $B \subset L$ there exists a constant m_B such that

$$|f(t, \varphi)| \leqslant m_B$$

for $t \in [a, b]$, $\varphi \in B$.

For a given family Φ of the functions $f\colon [a, b]\times L \to R^d$ we say that Φ satisfies *the Carathéodory conditions uniformly* if every function $f \in \Phi$ fulfils conditions (i), (ii) and

(iii') for every bounded set $B \subset L$ there exists a constant M_B such that

$$|f(t, \varphi)| \leqslant M_B$$

for every $t \in [a, b]$, $\varphi \in B$ and $f \in \Phi$.

We say that the sequence $\{f_n\}_{n=1}^{\infty}$ ($f_n\colon [a, b]\times L \to R^d$) *converges continuously* to a function $f\colon [a, b]\times L \to R^d$ if for every sequence $\{\varphi_n\} \subset L$ convergent to a function $\varphi \in L$ in the space L the sequence $\{f_n(t, \varphi_n)\}$ converges to $f(t, \varphi)$ for every $t \in [a, b]$ as $n \to \infty$.

Let us denote by $\mathscr{A}$ the space of all functions $f\colon [a, b]\times L \to R^d$ fulfilling the Carathéodory conditions (i), (ii), (iii).

We remind that the sequence $\{f_n\} \subset \mathscr{A}$ is *almost uniformly convergent* to a function $f \in \mathscr{A}$ if and only if $f_n(t, \varphi) \to f(t, \varphi)$ uniformly on $[a, b]\times B$ for every bounded set $B \subset L$.

In the sequel we shall not mention that the equality $\dot{x} = f(t, x_t)$ occurs almost everywhere. It will be clear from the context.

For $x\colon [a-h_0, b] \to R^d$ and $t \in [a, b]$ we define the function $x_t\colon [a-h_0, a] \to R^d$ by

$$x_t(\theta) = x(t+\theta-a)\,. \qquad (1)$$

Let f: $[a, b] \times L \to R^d$ be a function belonging to $\mathscr{A}$ and let $A \in \mathscr{L}$.

We shall consider the following boundary value problem

$$\dot{x}(t) = f(t, x_t), \tag{2}$$

$$Ax = \alpha. \tag{3}$$

We remind that an almost everywhere differentiable function x: $[a-h_0, b] \to R^d$ is said to be *a solution of equation* (2) if x satisfies (2) almost everywhere on $[a, b]$.

Condition (H^0). *The function* f: $[a, b] \times L \to R^d$ *fulfils condition* (H^0) if for every function $\varphi \in L$ there is exactly one solution x of equation (2) which satisfies the initial condition $x_a = \varphi$.

The continuous x: $[a-h_0, b] \to R^d$ is called *the solution of equation* (2) *in Carathéodory sense* if it is absolutely continuous on the interval $[a, b]$, satisfies (2) almost everywhere on $[a, b]$ and is constant on $[a-h_0, a]$, that is x_a = const.

The continuous function x: $[a-h_0, b] \to R^d$ is called *the solution of problem* (2), (3) if it is the solution of equation (2) in Carathéodory sense and fulfils condition (3).

3. Existence and uniqueness problems. Repeating without any essential modifications the proof of Hale's continuous dependence theorem ([3], p. 21—22) we get as a corollary.

THEOREM 1. *Let us assume that the functions* f_n: $[a, b] \times L \to R^d$ $(n = 0, 1, 2, \ldots)$ *satisfy the Carathéodory conditions uniformly and the sequence* $\{f_n\}_{n=1}^{\infty}$ *converges continuously to a function* f_0 *satisfying condition* (H^0). *Let the sequence* $\{\varphi_n\} \subset L$ *be convergent to* $\varphi_0 \in L$ *and let* x_n: $[a-h_0, b] \to R^d$ $(n = 0, 1, 2, \ldots)$ *be the solution of equation* $\dot{x}(t) = f_n(t, x_t)$ *satisfying the initial condition* $x_a = \varphi_n$. *Then* $x_n(t)$ *converges uniformly to* $x_0(t)$ *on* $[a-h_0, b]$.

By Theorem 1, using the same method as in proof of Theorem 3.1 ([5]) we get

THEOREM 2. *Let a fixed function* f: $[a, b] \times L \to R^d$ *satisfy the Carathéodory conditions and condition* (H^0). *Let* Ω *be an open set in* $\mathscr{L}$. *If for every operator* $A \in \Omega$ *and every vector* $\alpha \in R^d$ *boundary value problem* (2), (3) *has at most one solution, then for every* $A \in \Omega$ *and every* $\alpha \in R^d$ *problem* (2), (3) *has exactly one solution.*

Proof. Let $k \in R^d$. Denote by x^k the solution of (1) satisfying the initial condition $x_a^k = k$. It is sufficient to show that for every $A \in \Omega$ and $\alpha \in R^d$ there exists $k \in R^d$ such that $Ax^k = \alpha$.

Let $A^0 \in \Omega$ be given. Define the function F: $R^d \to R^d$ by the formula

$$F(k) = A^0 x^k.$$

It follows from condition (H^0), uniqueness assumption and Theorem 1 that F is a continuous injection. It remains to be shown that the range of F, $F(R^d)$, is in fact equal R^d.

Suppose that $F(R^d) \neq R^d$. Since $F(R^d)$ is open (by Schauder's open mapping theorem [2]), there exists a point $p \in R^d$ and a sequence $\{k_n\} \subset R^d$ such that

$$F(k_n) \to p \quad \text{as} \quad n \to \infty \tag{4}$$

and

$$p \notin F(R^d).$$

If there exists a bounded subsequence of $\{k_n\}$, then we can find a convergent subsequence $\{k_{n_i}\}_{i=1}^{\infty}$ of $\{k_n\}$. Let $k_{n_i} \to k_0$ as $n_i \to \infty$. Then $F(k_0) = p$ and we have a contradiction with the fact that $p \notin F(R^d)$. Therefore, we may assume that there exists $\varepsilon > 0$ such that

$$|k_{n+m} - k_n| \geqslant \varepsilon, \qquad n = 1, 2, 3, \dots; \qquad m = m(n) \geqslant 1.$$

We have

$$F(k_{n+m}) - F(k_n) = A^0(x^{k_{n+m}} - x^{k_n}) = A^0 y_n, \tag{5}$$

where $y_n = x^{k_{n+m}} - x^{k_n}$. Let $A^k\colon L \to R^d$ for $k = 1, 2, \dots$ be such that

$$A^k(y_n) = -A^0 y_n, \tag{6}$$

$$\|A^k\| = \frac{|A^0 y_n|}{\|y_n\|_L}. \tag{7}$$

The existence of these bounded linear operators satisfying (6) and (7) follows from Hahn-Banach theorem. Since

$$|y_n|_L \geqslant |(y_n)_a| = |x_a^{k_{n+m}} - x_a^{k_n}| = |k_{n+m} - k_n| \geqslant \varepsilon,$$

we have by (4) and (5) that

$$\|A^k\| \leqslant \frac{|A^0 y_n|}{\varepsilon} \to 0 \quad \text{as} \quad n \to \infty.$$

Since Ω is open, there exists $\bar{n}$ such that $A^0 + A^{\bar{n}} \in \Omega$. For $A^0 + A^{\bar{n}}$ we have

$$(A^{\bar{n}} + A^0)(x^{k_{\bar{n}+\bar{m}}} - x^{k_{\bar{n}}}) = 0, \qquad \bar{m} = m(\bar{n}).$$

This means that the boundary value problem

$$\dot{x}(t) = f(t, x_t),$$
$$(A^{\bar{n}} + A^0)x = (A^{\bar{n}} + A^0)x^{k_{\bar{n}}}$$

has two distinct solutions $x^{k_{\bar{n}+\bar{m}}}$ and $x^{k_{\bar{n}}}$. This contradicts our uniqueness assumption and completes the proof of the theorem.

Let $A_0 \in \mathscr{L}$ has the form

$$A_0 x = x(a). \tag{8}$$

It is obvious that boundary value problem (2), (4) is a particular case of the Cauchy problem. It is easy to see that if the function f in equation (2) fulfils condition (H^0), then the Cauchy problem for this equation has exactly one solution. Therefore, problem (2), (8) has the same property. We can show, using Theorem 2 and repeating the argumentation of the proof of Theorem 4.1 ([5]) that the problems close (in the sense of the topology in $\mathscr{L}$) to problem (2), (8) have exactly one solution. That is

THEOREM 3. *If a fixed function f: $[a, b]\times L \to R^d$ fulfils the Carathéodory conditions and the Lipschitz condition*

$$|f(t, \varphi)-f(t, \psi)| \leqslant K\|\varphi-\psi\|_{C^0}, \tag{9}$$

where K is a positive constant, then for every operator A belonging to the open ball $B(A_0, \eta)$, where

$$\eta = \frac{1+\exp(b-a)K}{(b-a)k} \tag{10}$$

and for every vector $\alpha \in R^d$ boundary value problem (2), (3) *has exactly one solution.*

Just in the same way as Theorem 6.1 ([5]) we can prove

THEOREM 4. *Let $\Phi \subset \mathscr{A}$ be a given family of functions f: $[a, b]\times L \to R^d$ fulfilling condition* (H^0) *and the Carathéodory conditions uniformly. Let Ω be an open set in $\mathscr{L}$. We assume that for every function $f \in \Phi$, for every operator $A \in \Omega$ and every vector $\alpha \in R^d$ boundary value problem* (2), (3) *has at most one solution. Then boundary value problem* (2), (3) *has exactly one solution dependent in a continuous manner upon $x(f, A, \alpha) \in \Phi \times \Omega \times R^d$, i.e., if $\{f_n\} \subset \Phi$ is almost uniformly convergent to $f \in \Phi$, $\{A_n\} \subset \Omega$ is convergent to $A \in \Omega$* (in $\mathscr{L}$), *$\{\alpha_n\} \subset R^d$ is convergent to $\alpha \in R^d$, then $x(f_n, A_n, \alpha_n) \to x(f, A, \alpha)$ as $n \to \infty$, in the space W.*

4. Application. Let a fixed function f: $[a, b]\times R^d \to R^d$ be given. We define functions $\bar{f}$: $[a, b]\times L([2a-b, a], R^d) \to R^d$ and $\bar{f}_n$: $[a, b]\times L([2a-b, a], R^d) \to R^d$ by the formulas

$$\bar{f}(t, \varphi) = f(t, \varphi(a)), \tag{11}$$

$$\bar{f}_n(t, \varphi) = f\big(\sigma^n(t), \varphi(\sigma^n(t)-t+a)\big), \tag{12^n}$$

where

$$\sigma^n(t) = \max_k \left\{\frac{b-a}{n} k < t\right\}.$$

Let us consider boundary value problems

$$\dot{x}(t) = \bar{f}(t, x_t), \qquad Ax = \alpha; \tag{13}$$

$$\dot{x}(t) = \bar{f}_n(t, x_t), \qquad A_n x = \alpha_n, \tag{14^n}$$

where the sequence of operators $\{A_n\}_{n=1}^{\infty}$ is convergent to an operator A in the space $\mathscr{L}$ and the sequence of vectors $\{\alpha_n\}_{n=1}^{\infty}$ is convergent to a vector α in the space R^d.