UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXVI

1987

The Newton-Padé Approximants and the Growth of Meromorphic Functions

by Krzysztof RECZEK

1. Introduction. Let f be a function holomorphic in a set $\{z: |z| > R\}$. To estimate the growth of the modulus of $f(z)$ when z tends to infinity, we introduce the order and the type of the function f.

Denote $M_f(r) = \sup\{|f(z)|: |z| = r\}$. Then we define the *order* of f as follows:

$$\varrho(f) = \varrho = \inf\{\mu > 0: \exists r_0 \forall r > r_0 M_f(r) < \exp(r^\mu)\}.$$

When $0 < \varrho < \infty$, we define the *type* of f in the following way:

$$\sigma(f) = \sigma = \inf\{v > 0: \exists r_0 \forall r > r_0 M_f(r) < \exp(vr^\varrho)\}.$$

Suppose now that f is an entire function. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then

(1.1) $$\varrho(f) = \limsup_{n\to\infty} \frac{-n\ln n}{\ln|a_n|}$$

and

(1.2) $$\sigma(f) = \limsup_{n\to\infty} \frac{n|a_n|^{\varrho/h}}{e\varrho}$$

if $0 < \varrho < \infty$ (see [2]).

Let $(z_n)_{n=1}^{\infty}$ be a bounded sequence of points. Denote

(1.3) $$\omega_0(z) = 1, \quad \omega_n(z) = (z - z_n)\cdot\omega_{n-1}(z), \quad n = 1, 2, 3 \ldots$$

Let f be an entire function. Consider the Newton development of f with respect to the sequence (z_n):

(1.4) $$f(z) = \sum_{n=0}^{\infty} c_n \omega_n(z).$$

Winiarski [4] has proved that the order and the type of f are given by the formulas:

(1.5) $$\varrho(f) = \limsup_{n\to\infty} \frac{-n\ln n}{\ln|c_n|},$$

(1.6) $$\sigma(f) = \limsup_{n\to\infty} \frac{n|c_n|^{\varrho/h_j}}{e\varrho}, \quad 0<\varrho<\infty,$$

which are generalizations of the classical formulas (1.1) and (1.2). Moreover, he has shown that if a sequence (c_n) satisfies the condition

$$\limsup_{n\to\infty} n^{1/\varrho}|c_n|^{1/n} = (e\varrho\sigma)^{1/\varrho}, \quad \varrho>0,\ \sigma>0,$$

then the function f defined by (1.4) is an entire function of order ϱ and of type σ.

The purpose of this paper is to give some generalizations of the above formulas for the functions meromorphic in $\boldsymbol{C}$ with a finite number of poles. In order to do this, we need to replace the Newton coefficients by the coefficients of the Newton-Padé approximants.

2. The Newton-Padé approximants. Let $(z_n)_{n=1}^{\infty}$ be a sequence of complex numbers. Suppose that f is a function holomorphic in a neighbourhood of the set $\{z_n\colon 1\leqslant n<\infty\}$.

Denote by $R_{n,m}$ the set of all rational functions, whose numerators and denominators are polynomials of degrees not greater than n and m, respectively. Let the function $f_{n,m}$ satisfy the following conditions:

1° $f_{n,m}\in R_{n,m}$;

2° the function $\dfrac{f-f_{n,m}}{\omega_{n+m+1}}$ is holomorphic at each point z_i for $1\leqslant i\leqslant n+m+1$.

For each couple (n, m) there exists at most one function satisfying the above conditions. It is called the (n, m)-th *Newton-Padé approximant* of the function f with respect to the sequence $(z_n)_{n=1}^{\infty}$ (see [1]).

In the sequel we will consider the sequences of Newton-Padé approximants $(f_{n,m})$ with m fixed and with n tending to infinity. It will be useful to simplify the notations. Denote:

(2.1) $$f_n := f_{n,m} = \frac{p_n}{q_n},$$

where

(2.2) $$p_n(z) = \sum_{i=0}^{n} p_{ni} z^i$$

and

(2.3) $$q_n(z) = (z-z_{n,1})\cdot\ldots\cdot(z-z_{n,m_n}),$$

where $z_{n,1}, \ldots, z_{n,m_n}$ are poles of the approximant f_n. Then the polynomials p_n and q_n have no common divisors of degree higher than zero. Assume that

$$|z_{n,1}| \leqslant \ldots \leqslant |z_{n,m_n}|. \tag{2.4}$$

We will use the following

LEMMA 2.1. *Let $(z_n)_{n=1}^{\infty}$ be a bounded sequence of complex numbers and let f be a function meromorphic in* $\mathbf{C}$, *holomorphic in a neighbourhood of the set $\{z_n: 1 \leqslant n < \infty\}$. Suppose that f has exactly m poles in* $\mathbf{C}$, *counted with their multiplicities. Then*:

1° *for almost every n there exists the approximant f_n*;

2° *the poles of f_n tend to the poles of f when n tends to infinity*;

3° $\lim_{n\to\infty} f_n(z) = f(z)$ *in* $\mathbf{C}$, *except for the poles of f.*

This lemma is a slight modification of the Saff theorem [3], so we omit the proof.

3. The growth of meromorphic functions. Denote by $M_m(C)$ the class of all functions meromorphic in C, whose number of poles is not greater than m. The main result of this paper is the following

THEOREM 3.1. *Let $(z_n)_{n=1}^{\infty}$ be a bounded sequence of complex numbers. Let Ω be a domain containing the set $\{z_n: 1 \leqslant n < \infty\}$. Assume that there exists a limit point of the sequence (z_n) in Ω. Let f be a function meromorphic in Ω and holomorphic at each point z_n for $1 \leqslant n < \infty$. Assume that for almost every n there exists the (n, m)-th Newton-Padé approximant f_n with respect to the sequence $(z_n)_{n=1}^{\infty}$ and that for some positive numbers μ and ν the following inequality is true*:

$$\limsup_{n\to\infty} n^{1/\mu} |p_{nn}|^{1/n} \leqslant (e\mu\nu)^{1/\mu}. \tag{3.1}$$

Then:

1° *f can be extended to a function of the class $M_m(\mathbf{C})$ (we will denote this meromorphic continuation by the same letter f)*;

2° *the order of f is not greater than μ*;

3° *if $\varrho(f) = \mu$ then the type of f is not greater than ν*;

4° *if*

$$\limsup_{n\to\infty} n^{1/\mu} |p_{nn}|^{1/n} = (e\mu\nu)^{1/\mu} \tag{3.1'}$$

and if

$$\limsup_{n\to\infty} |z_{n,m_n}|^{1/n} \leqslant 1, \tag{3.2}$$

then $\varrho(f) = \mu$ and $\sigma(f) = \nu$.

Proof. Suppose that there exist the approximants f_n for $n \geq n_0$. Choose a number $\theta \in (0, 1)$. We define the set D_θ as the sum

$$D_\theta = \bigcup_{n=n_0}^{\infty} \bigcup_{i=1}^{m_n} B(z_{n,i}, \theta^n), \tag{3.3}$$

where $B(a, r) = \{z: |z-a| < r\}$. Hence, the set D_θ can be covered by a number of discs, whose sum of diameters is not greater than $d_\theta := \frac{2m}{1-\theta}$. Moreover, if $z \in C \setminus D_\theta$, then it follows from (2.3) and (3.3) that

$$|q_n(z)| \geq \theta^{mn}. \tag{3.4}$$

Put $s = \sup\{|z_n|: 1 \leq n < \infty\}$. Let $z \in C \setminus D_\theta$. It follows from (2.1), (2.2) and (2.3) that

$$f_n(z) - f_{n-1}(z) = \frac{p_{nn}\omega_{n+m}(z)}{q_{n-1}(z)q_n(z)}.$$

Hence and from (3.4) we get

$$|f_n(z) - f_{n-1}(z)| \leq \theta^{-2mn}|p_{nn}|(|z|+s)^{m+n} \tag{3.5}$$

when $z \in C \setminus D_\theta$.

But $\lim_{n\to\infty} |p_{nn}|^{1/n} = 0$. Hence the sequence $(f_n(z))_{n=n_0}^{\infty}$ converges if only z does not belong to D_θ.

Choose a number $R > s$. Then there exists R_θ, $R \leq R_\theta < R + d_\theta$, such that the set D_θ does not intersect the circle $C(0, R_\theta) = \{z: |z| = R_\theta\}$. Obviously, the sequence $(f_n)_{n=n_0}^{\infty}$ is uniformly convergent on $C(0, R_\theta)$.

For every n let $k_n = \max\{i \leq m_n: |z_{n,i}| \leq R_\theta\}$. Denote

$$Q_n(z) = (z - z_{n,1}) \cdot \ldots \cdot (z - z_{n,k_n}).$$

Then $|Q_n(\zeta)| \leq (2R_\theta)^m$ for $|\zeta| \leq R_\theta$. Hence we can choose an increasing sequence of integers (n_k) such that the sequence $(Q_{n_k})_{k=1}^{\infty}$ converges in C. Put $Q = \lim_{n\to\infty} Q_{n_k}$. Then Q is a polynomial of degree not greater than m.

Each function $f_n \cdot Q_n$ is holomorphic in a neighbourhood of the disc $B(0, R_\theta)$. Hence the sequence $(f_{n_k} \cdot Q_{n_k})_{k=1}^{\infty}$ converges to a function φ holomorphic in the set $B(0, R_\theta)$. Moreover, it follows from the identity principle that $\varphi = f \cdot Q$ in $\Omega \cap B(0, R_\theta)$. Hence, the function f can be extended to the function $\frac{\varphi}{Q}$, meromorphic in the set $B(0, R_\theta)$, and the number of poles of this function lying in that disc is not greater than m. But R has been chosen arbitrarily, so f can be extended to a function of the class $M_m(C)$. We denote it by the same letter f. This proves 1°.

Before we prove 2°, we have to make some additional remarks. Let $z \in C \setminus D_\theta$. Suppose that there exists a sequence (n_l) and a neighbourhood U of the point z such that for every l the function f_{n_l} has no poles in U. Then it can be shown in the previous way that

$\lim_{n\to\infty} f_n(z) = \lim_{l\to\infty} f_{n_l}(z) = f(z)$. So we have shown that $\lim_{n\to\infty} f_n(z) = f(z)$ in $C\setminus D_\theta$ except for at most m points. We can choose a number R_0 such that for every point

$$z \in (C\setminus D_\theta)\setminus B(0, R_0), \ \lim_{n\to 0} f_n(z) = f(z).$$

Assume that R_0 is so great that $M_f(R)$ is an increasing function for R larger than R_0.

Let R be greater than R_0. Then

$$M_f(R) \leqslant M_f(R_\theta) \leqslant \|f_{no}\|\, C(0, R_\theta) + \sum_{n=n_0}^{\infty} \|f_n - f_{n-1}\|\, C(0, R_\theta).$$

According to (3.5), we have

(3.6) $$M_f(R) \leqslant A_1 (R_\theta)^{no} + \sum_{n=n_0+1}^{\infty} |p_{nn}| \theta^{-2mn} (R_\theta + s)^{m+n}.$$

Let K be an arbitrary number greater than ν. Then it follows from (3.1) that there exists a number $n_1 \geqslant n_0$ such that

$$|p_{nn}| \leqslant \left(\frac{eK\mu}{n}\right)^{n/\mu} \quad \text{for } n \geqslant n_1.$$

If R is large enough, then $(R_\theta + s) \leqslant \theta^{-m} \cdot R$. It follows from (3.5) and (3.6) that

$$M_f(R) \leqslant A_2 R^{n_1} + (\theta^{-m} \cdot R)^m \sum_{n=n_1+1}^{\infty} \left(\frac{eK\mu}{n}\right)^{n/\mu} (\theta^{-3m} \cdot R)^n,$$

where A_2 depends only on θ.

Let n_R be the smallest integer greater than $2^\mu eK\mu(\theta^{-3m} \cdot R)^\mu$. Then n_R is greater than n_1, if R is large enough, and the sum $\sum_{n \geqslant n_R} |p_{nn}| (\theta^{-3m} \cdot R)^n$ is smaller than 1. Consequently,

(3.7) $$M_f(R) \leqslant A_2 R^{n_1} + (\theta^{-m} \cdot R)^m \cdot \left(n_R \cdot \max\left\{\left(\frac{eK\mu}{n}\right)^{n/\mu} \cdot (\theta^{-3m} \cdot R)^n\right\} + 1\right),$$

when R is large enough. From (3.7) we get

$$M_f(R) \leqslant A_2 R + (\theta^{-m} \cdot R)^m \cdot A_3 R^\mu \cdot \exp\big(K(\theta^{-3m} \cdot R)^\mu\big),$$

where A_3 depends only on θ, μ and K. This implies that the order of f is not greater than μ and if $\varrho(f) = \mu$, then the type of f does not exceed $K\theta^{-3m\mu}$. But θ can be chosen as close to 1 as we need, so the type of f is not greater than Ω and, consequently, not greater than ν. This proves 2° and 3°.

Assume now that the conditions (3.1′) and (3.2) are satisfied. Then, of course, f can be extended to a function of the class $M_m(C)$. Then we can write $f = \dfrac{\varphi}{Q}$, where φ is an entire function and Q is a polynomial of the form

(3.8) $$Q(z) = (t - \zeta_1) \cdot \ldots \cdot (z - \zeta_k),$$

where k is the number poles of f. Then, of course, the order of φ is equal to the order of f and the type of φ is equal to the type of f.

For the proof of 4°, assume that either the order of f is smaller than μ or the type of f is smaller than ν. Then there exists a number $K < \nu$, such that

$$|\varphi(z)| \leqslant \exp(K|z|^{\mu}), \tag{3.9}$$

when $|z|$ is large enough. Using the Cauchy formula we get from (2.2) and (3.8)

$$p_{nn} = \frac{1}{2\pi i} \int\limits_{C(0,r)} \frac{\varphi(z) q_n(z)}{\omega_{m+n+1}(z)} dz$$

for $r > s$. Using (2.2), (2.3) and (3.9) we obtain the estimation

$$|p_{nn}| \leqslant \frac{r \cdot 2^m (r^m + |z_{n,m_n}|^m) \exp(Kr^{\mu})}{\min\{|\omega_{m+n+1}(z)|:\ |z| = r\}}, \tag{3.10}$$

When r is large enough. Put $r = \left(\frac{n}{K\mu}\right)^{1/\mu}$. Then for almost every n the estimation (3.10) is true. Hence we derive

$$\limsup_{n\to\infty} n^{1/\mu} |p_{nn}|^{1/n} \leqslant (eK\mu)^{1/\mu} < (e\mu\nu)^{1/\mu}.$$

This contradicts the assumed equality (3.1′). We have proved 4°.

As consequences of theorem 3.1, we derive the following corollaries:

COROLLARY 3.1. *If* $\lim_{n\to\infty} n^{\alpha} |p_{nn}|^{1/n} = 0$ *for every positive* α, *then* f *is a function of order zero.*

COROLLARY 3.2. *Assume that the function* f *has exactly* m *poles in* $\mathbf{C}$ *and that the equality* (3.1′) *is satisfied. Then the order of* f *is equal to* μ *and the type of* f *is equal to* ν.

Proof. Lemma 2.1 implies that the approximants f_n satisfy the condition (3.2).

References

[1] M. A. Gallucci, W. B. Jones, *Rational approximations corresponding to Newton series*, J. Approx. Theory 17 (1976), 366—392.

[2] А. И. Маркушевич, *Теория аналитических функций*, Москва, 1957.

[3] E. B. Saff, *An extension of Montessus de Ballore's theorem on the convergence of interpolating rational functions*, J. Approx. Theory 6 (1972), 63—67.

[4] T. Winiarski, *Approximation and interpolation of entire functions*, Ann. Polon. Math. 23 (1970), 259—273.

Received February 3, 1982.