

Another Generalization of Strong Unicity

Jan SUDOLSKI and Adam WÓJCIK

Abstract. In this note another concept of generalization of strong unicity is considered. Some properties and examples are given.

1. Introduction. The notion of strong unicity of best approximation was introduced by Newman and Shapiro, who proved the following

THEOREM 1.1, [11]. *Given a Haar subspace V of the Banach space $C[0, 1]$ of real- or complex-valued continuous functions on the interval $[0, 1]$ and an element $f \in C[0, 1]$, there exist $g \in V$ and a constant $r > 0$ such that for every $h \in V$*

$$(1.1) \quad \|f-h\| \geq \|f-g\| + r\|h-g\|$$

in the real case or

$$(1.2) \quad \|f-h\|^2 \geq \|f-g\|^2 + r\|h-g\|^2$$

in the complex one.

In the real case the element g satisfying (1.1) is called the strongly unique element of best approximation for f with respect to V . The strong unicity problem can be considered in an arbitrary normed linear space but, by Wulbert's remark [17], in a smooth space E there are no strongly unique elements of best approximation in the above sense with respect to any subspace of E .

The first concept of generalization of the strong unicity was given by McLaughlin and Somers [10]. If E is a real normed linear space and V is its subspace then, for a given $f \in E$; denote by $P_V(f)$ the set of all elements of best approximation to f in V . The strong unicity inequality is defined as follows: there exists $r > 0$ such that for every $h \in V$

$$(1.3) \quad \|f-h\| \geq \text{dist}(f, V) + r \text{dist}(h, P_V(f)).$$

In the real space $C[0, 1]$ the only subspaces with the property that for every $f \in C[0, 1]$ the inequality (1.3) holds are Haar subspaces. In general, (1.3) can be satisfied even in the case where V is not a Chebyshev subspace.

The aim of our paper is to generalize the notion of strong unicity in another way to include both real and complex cases of Theorem 1.1. Most of important properties of

the strongly unique elements of best approximation are preserved and, as is shown by examples, this notion can be discussed in Hilbert spaces and L^p spaces. Notions which are not defined in the paper can be found in [13].

2. Definitions. Throughout the paper E will denote a normed linear space and G its closed subset.

DEFINITION 2.1. Let f be an element of the space E and $p \geq 1$. An element $g \in G$ is called *the p -strongly unique element of best approximation for f with respect to G* (briefly: p -SUBA) iff there exists a positive constant r such that for every $h \in G$ the following inequality holds

$$(2.1) \quad \|f-h\|^p \geq \|f-g\|^p + r\|g-h\|^p.$$

It is clear that if g is p -SUBA for f (with any p) then g is also the unique element of best approximation for f with respect to G .

Remark 2.2. If g is p -SUBA for f and $q > p$ then g is also q -SUBA for f .

DEFINITION 2.3. If for every $f \in E$ there exists p -SUBA with respect to G then G is called *the strongly Chebyshev set* (briefly: SCS). If p does not depend on f , then G is called *the p -strongly Chebyshev set* (briefly: p -SCS).

Observe that Theorem 1.1 can be reformulated as follows:

Every Haar subspace of $C[0, 1]$ is a p -SCS, where $p = 1$ in the real case and $p = 2$ in the complex case.

3. Characterizations. For $x \in E$ and $d > 0$, let $\bar{B}(x, d)$ denote the closed ball $\{y \in E: \|x-y\| \leq d\}$ and $B(x, d)$ the open ball $\{y \in E: \|x-y\| < d\}$. In the sequel we assume that $f \in E \setminus G$ and $g \in G$.

We start with some conditions equivalent to (2.1). We need two lemmas.

LEMMA 3.1. Let $p, q \geq 1$ and $d > 0$. The following conditions are equivalent: there exists $r > 0$ such that for every $h \in G \cap B(g, d)$ the inequality

$$(3.1) \quad \|f-h\| \geq \|f-g\| + r\|g-h\|^p$$

holds,

there exists $r' > 0$ such that for every $h \in G \cap B(g, d)$ the inequality

$$(3.2) \quad \|f-h\|^q \geq \|f-g\|^q + r'\|g-h\|^p$$

holds.

Proof. It is clear that if $B < A$ then

$$qB^{q-1}(A-B) \leq A^q - B^q \leq qA^{q-1}(A-B).$$

Hence by (3.1) we obtain

$$\|f-h\|^q - \|f-g\|^q \geq q\|f-g\|^{q-1}(\|f-h\| - \|f-g\|) \geq rq\|f-g\|^{q-1}\|g-h\|^p.$$

Conversely, by (3.2)

$$\|f-h\| - \|f-g\| \geq \frac{r'}{q\|f-h\|^{q-1}}\|g-h\|^p \geq \frac{r'}{q(\|f-g\|+d)^{q-1}}\|g-h\|^p.$$

LEMMA 3.2. *If G is boundedly compact and g is the unique element of best approximation for f in G then for every $q \geq 1$ and every $d > 0$ there exists a positive constant r such that for every $h \in G \setminus B(g, d)$*

$$(3.3) \quad \|f-h\|^q \geq \|f-g\|^q + r\|h-g\|^q.$$

Proof. Fix $q \geq 1$ and $d > 0$. Let us consider the continuous positive function

$$F(h) := (\|f-h\|^q - \|f-g\|^q) \cdot \|g-h\|^{-q}$$

defined on $G \setminus B(g, d)$. If G is bounded then the set $K := G \setminus B(g, d)$ is compact and F attains its minimum m on K , whence (3.3) is satisfied with $r = m$. If G is unbounded, it is evident that

$$\lim_{\|g-h\| \rightarrow \infty} F(h) = 1.$$

Hence there exists $M > 0$ such that $F(h) \geq \frac{1}{2}$ for $h \in G \setminus B(g, M)$. Then (3.3) holds for $r = \min(n, 1/2)$ where n is the minimum value of F on the compact set

$$\{h \in G : d \leq \|h-g\| \leq M\}.$$

For a fixed number $p \geq 1$, put

$$R_p(t) := \begin{cases} t^p, & \text{if } 0 \leq t \leq 1 \\ t, & \text{if } t > 1. \end{cases}$$

PROPOSITION 3.3. *Let $f \in E \setminus G$ and $g \in G$. The following statements are equivalent:*

(i) *there exists $r_1 > 0$ such that for every $h \in G$*

$$\|f-h\|^p \geq \|f-g\|^p + r_1\|g-h\|^p,$$

(ii) *for every $d > 0$ there exists $r_2 > 0$ such that for every $h \in G \cap \bar{B}(g, d)$*

$$\|f-h\| \geq \|f-g\| + r_2\|g-h\|^p,$$

(iii) *there exists $r_3 > 0$ such that for every $h \in G$*

$$\|f-h\| \geq \|f-g\| + r_3 R_p\left(\frac{\|g-h\|}{\|f-g\|}\right),$$

(iv) *there exists $r_4 > 0$ such that for every $h \in G$*

$$\|f-h\|^p \geq \|f-g\|^p + r_4 R_p\left(\frac{\|g-h\|}{\|f-g\|}\right).$$

If moreover

(a) G is starlike with respect to g

or

(b) G is boundedly compact and g is the unique element of best approximation for f in G then (i)—(iv) are equivalent to each of the following conditions:

(v) there exist $D > 0$ and $r_5 > 0$ such that for every $h \in G \cap \bar{B}(g, D)$

$$\|f-h\| \geq \|f-g\| + r_5 \|g-h\|^p,$$

(vi) there exist $D > 0$ and $r_6 > 0$ such that for every $h \in G \cap \bar{B}(g, D)$

$$\|f-h\|^p \geq \|f-g\|^p + r_6 \|g-h\|^p.$$

Proof. (i) \Rightarrow (ii) follows from Lemma 3.1.

(ii) \Rightarrow (iii). By the argument of the proof of Lemma 3.2 there exists $d > \|f-g\|$ such that for every $h \in G \setminus \bar{B}(g, d)$

$$\|f-h\| - \|f-g\| \geq \frac{1}{2} \|g-h\| = \frac{1}{2} \|f-g\| R_p \left(\frac{\|g-h\|}{\|f-g\|} \right).$$

By (ii), for every $h \in G \cap \bar{B}(g, d)$

$$\|f-h\| - \|f-g\| \geq r_2 \|f-g\|^p R_p \left(\frac{\|g-h\|}{\|f-g\|} \right).$$

Hence (iii) holds with $r_3 = \min(\frac{1}{2} \|f-g\|, r_2 \|f-g\|^p)$.

(iii) \Rightarrow (iv) follows from the first part of the proof of Lemma 3.1.

(iv) \Rightarrow (i). Again by the argument of the proof of Lemma 3.2 there exists $M > \|f-g\|$ such that for every $h \in G \setminus \bar{B}(g, M)$

$$\|f-h\|^p - \|f-g\|^p \geq \frac{1}{2} \|g-h\|^p.$$

Let $h \in G \cap \bar{B}(g, M)$. If $\|g-h\| \leq \|f-g\|$ then

$$\|f-h\|^p - \|f-g\|^p \geq \frac{r_4}{\|f-g\|^p} \|g-h\|^p.$$

If $\|g-h\| \geq \|f-g\|$ then

$$\|f-h\|^p - \|f-g\|^p \geq \frac{r_4}{\|f-g\| \|g-h\|^{p-1}} \|g-h\|^p \geq \frac{r_4}{\|f-g\| M^{p-1}} \|g-h\|^p.$$

Thus we can take $r_1 = \min(1/2, r_4/M^p)$.

(v) \Leftrightarrow (vi) follows from Lemma 3.1.

(i) \Rightarrow (vi) is obvious.

(a) and (v) \Rightarrow (ii). Fix $d > D$ and take $h \in G \cap \bar{B}(g, d)$. Since G is starlike,

$$h' := g + \frac{D}{d} (h-g) \in G \quad \text{and} \quad \|h'-g\| \leq D.$$

Hence

$$\begin{aligned} r_5 \|g-h'\|^p &\leq \|f-h'\| - \|f-g\| = \left\| \left(1 - \frac{D}{d}\right)(f-g) + \frac{D}{d}(f-h) \right\| - \|f-g\| \leq \\ &\leq \frac{D}{d} (\|f-h\| - \|f-g\|). \end{aligned}$$

So we obtain

$$r_5 \left(\frac{D}{d}\right)^{p-1} \|h-g\|^p \leq \|f-h\| - \|f-g\|$$

what gives (ii).

(b) and (vi) \Rightarrow (i) follows from Lemma 3.2.

Remark 3.4. By a slight modification of the well known Kolmogorov criterion [6] we obtain the p -strong Kolmogorov criterion: if for $f \in E \setminus G$ and $g \in G$ there exists a positive constant r such that for every $h \in G$

$$(3.4) \quad \sup\{\operatorname{Re}L(g-h): L \in M(f-g)\} \geq r R_p\left(\frac{\|g-h\|}{\|f-g\|}\right),$$

where $M(f-g) := \{L \in E^*: \|L\| = 1, L(f-g) = \|f-g\|\}$, then g is p -SUBA for f with respect to G .

By [16], if $p = 1$, and G is starlike with respect to g then (3.4) is also a necessary condition for the strong uniqueness. On the other hand, for $p > 1$ the necessity of (3.4) fails to hold in a large class of spaces (e.g. if E is smooth and G is its subspace then by Theorem I.1.1, [13], the left-hand side of (3.4) is less than or equal to zero while, as is shown in Examples 5.3 and 5.4, p -strongly unique elements of best approximation can occur).

We define (set-valued) metric projection onto G as the mapping

$$P_G: E \ni x \rightarrow P_G(x) = \{z \in G: \|x-z\| = \operatorname{dist}(x, G)\}.$$

If G is a Chebyshev set then P_G is a vector-valued function.

Problem 3.5. Given $f \in E \setminus G$ and $g \in P_G(f)$, find a necessary and sufficient condition on g to be the p -strongly unique element of best approximation to f in G (for some p).

4. Properties. In Theorem 4.2 some basic properties of strongly Chebyshev sets are gathered. It is seen that such sets must be "very good" and that the property "to be a strongly Chebyshev set" is rather rare, e.g. the only strongly Chebyshev subspaces of $C[a, b]$ are Haar subspaces (see [9]) and the only strongly Chebyshev sets in a Hilbert space are closed convex subsets (see Corollary 5.5). The following lemma is a generalization of the known theorem of Freud (see, e.g. [12], Corollary 2.4.7).

LEMMA 4.1. Let G be a closed subset of E and $f \in E \setminus G$: If g is p -SUBA for f with respect to G then there exist a neighbourhood U of f and a constant M such that for every $k \in U$ and $h \in P_G(k)$

$$\|g-h\| \leq M \|f-k\|^{1/p}.$$

