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On some Properties of Solutions to the First Fourier Problem for Infinite System of Parabolic Differential-functional Equations in an Arbitrary Domain

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1. Introduction. In this paper we shall consider the so-called first Fourier problem for a weakly coupled infinite system of parabolic differential-functional equations

$$u_t^i(t,x) = f^i(t,x,u(t,x),u_x^i(t,x),u_{xx}^i(t,x),u) \quad i = 1,2,\ldots \tag{1.1}$$

(where $(t,x) \in D$, D-infinite domain, $u = (u^1, u^2, \ldots)$, $u_x^i = (u_{x_1}^i, \ldots, u_{x_1}^i)$ and u_{xx}^i is the matrix formed by the second order derivatives $u_{x_j x_k}^i$) and for a strongly coupled infinite system of parabolic equations

$$u_t^i(t,x) = f^i(t,x,u(t,x)\ u_{x_n}(t,x),u_x^i(t,x),\ u_{xx}^i(t,x)) \quad i = 1,2,\ldots \tag{1.2}$$

(where $u_{x_n} = (u_{x_n}^1, u_{x_n}^2, \ldots)$).

Our concern will be to prove some theorems (comparison theorem, theorem of the uniqueness of the solution and theorem of the continuous dependence of the solution upon the right hand sides of the system and upon the initial and boundary conditions) for a solution of system (1.1) or (1.2) belonging to the functional class $E_2^{1,\infty}$ (for a definition of the class — see below) and satisfying given initial and boundary conditions. Similar problems for a solution $u \in l^\infty$ were elaborated in [3] by J. Szarski.

In order to make further considerations more clear let us introduce some definitions, notations and special general assumptions.

Let D be a set in time-space $(t,x) = (t, x_1, \ldots, x_n)$. We shall say that the set D has the property P if

1°. the projection of the interior of D on t-axis is the interval (t_0, t_0+T), where t_0 is finite and $0 < T < \infty$,

2°. for every $(\tilde{t}, \tilde{x}) \in D$ there exists a positive number r such that

$$\{(t,x):\ (t-\tilde{t})^2 + \sum_{j=1}^{n} (x_j - \tilde{x}_j)^2 < r^2,\ t \leqslant \tilde{t}\} \subset D.$$

By D_0 we denote the set which fulfils the following conditions:

$$D_0 \supset \bar{D},\ D_0 \subset \{(t,x):\ t \leqslant t_0+T,\ x \in \mathscr{R}^n\}.$$

Let us fix $K>0$, $M>0$ and $l\in\{1,2,\dots,n\}$. By $E_2^l(K, M; D_0)$ in short E_2^l let us denote the class of functions u defined in set D_0 for which the inequality

$$|u(t,x)| \leqslant M\exp\Big(\sum_{j=1}^{l} x_j^2\Big).$$

holds true in D_0.

Whereas by $E_2^{l,\infty}(\Omega, M; D_0)$ (in short $E_2^{l,\infty}$) let us denote the real space of mappings

$$w = (w^1, w^2, \dots)\colon\ D_0 \ni (t,x) \to w(t,x) = \big(w^1(t,x), w^2(t,x), \dots\big) \in l^\infty$$

where w^i are continuous in D and belong to E_2^l, and l^∞ is the real Banach space of sequences $s = (s^1, s^2, \dots)$ for which there exists a finite norm $||s||_{l^\infty} = \sup\{|s^v|\colon v = 1,2,\dots\}$. In class $E_2^{l,\infty}$ we introduce the norm

$$||w||^l = \sup\{|w^j(t,x)|\exp(-K\sum_{j=1}^{l} x_j^2)\colon (t,x)\in D_0,\ j = 1,2,\dots\}.$$

For $w\in E_2^{l,\infty}$ and fixed $t\geqslant t_0$ we define

$$||w||_t^l = \sup\{|w^j(\tilde t,\tilde x)|\exp(-K\sum_{j=1}^{l}\tilde x_j^2)\colon (\tilde t,\tilde x)\in D,\ \tilde t\leqslant t,\ j = 1,2,\dots\}.$$

We say that the mapping $u = (u^1, u^2, \dots)\in E_2^{l,\infty}$ is regular if for every $i\in N$ functions u are continuous in $\bar D$ and u_t^i, u_x^i, u_{xx}^i are continuous in D.

Let us introduce.

Assumptions A. Real functions $f^i(t,x,z,q,r,w)$ $(i = 1,2,\dots)$ are defined for $(t,x)\in D$, $z = (z_1, z_2, \dots)\in l^\infty$, $q = (q_1,\dots,q_n)\in\mathscr{R}^n$, $r = (r_{jk})_{j,k=1}^n$, $w\in E_2^{l,\infty}$. We say that a regular solution of (11) is parabolic if for every matrix $r\in\mathscr{R}^{n\times n}$ and $\eta^2 = 1$ the following implication

$$r\geqslant 0 \Rightarrow \eta f^i(t,x,u(t,x),u_x^i(t,x),u_{xx}^i(t,x)+\eta r, u)$$
$$\geqslant \eta f^i(t,x,u(t,x),u_x^i(t,x),u_{xx}^i(t,x),u)$$

holds true for $(t,x)\in D$ and $i = 1,2,\dots$

We say that a function

$$\varphi\colon \mathscr{R}\to\mathscr{R}$$

is of the class $\mathscr{L}(\kappa)$, where κ is a positive constant, if

1°. derivatives $\varphi^{(j)}$ $(j = 1,2,\dots)$ do exist and are continuous in $\mathscr{R}$,

2°. $||\varphi|| = \sup\{|\varphi(x)|\colon x\in\mathscr{R}\} < \infty$,

3° there exists a constant $c\in\mathscr{R}$ such that

$$||\varphi^{(k)}|| = \sup\{|\varphi^{(k)}(x)|\colon x\in\mathscr{R}\}\leqslant \varkappa^k c \quad \text{for } k = 1,2,\dots$$

For two functions $\varphi_1, \varphi_2\in\mathscr{L}(\varkappa)$ the inequality

(1.3) $$||\varphi_1' - \varphi_2'|| \leqslant \kappa||\varphi_1 - \varphi_2|| \quad \text{holds true [1].}$$

In the case of $D = D_{n-1} \times \mathscr{R}$, where $D_{n-1} \subset \{(t, x_1, \ldots, x_{n-1}) \in \mathscr{R}^n\}$ we shall say that a function

$$\psi = (\psi_1, \psi_2, \ldots)\colon D \ni (t, x) \to (\psi_1(t, x), \psi_2(t, x), \ldots)$$

belongs to the class $\mathscr{L}(\varkappa, x_n; D)$ if for every established $(t, x_1, \ldots, x_{n-1})$ and $j = 1, 2, \ldots$ function ψ_j as a function of variable x_n belongs to class $\mathscr{L}(\varkappa)$.

2. Theorems for a weakly coupled infinite system of parabolic differential-functional equations. In order to prove the comparison theorem given below we shall use the following lemma which is a consequence of theorem 1 of paper [2] and which was formulated by J. Szarski [3].

LEMMA. *Let D^* have the property P and let $D_0^* \supset D^*$. Let us suppose the real function $f(t, x, y, q, r)$ is defined for $(t, x) \in D^*$, $y \in \mathscr{R}$, $q \in \mathscr{R}^n$, $r \in \mathscr{R}^{n\times n}$ and satisfies the Lipschitz condition*

(2.1) $$[f(t, x, y, q, r) - f(t, x, \tilde{y}, \tilde{q}, \tilde{r})]\operatorname{sgn}(y-\tilde{y}) \leqslant L\Big(|y-\tilde{y}| + \sum_{j=1}^{n} |q_j - \bar{q}_j| + \sum_{j,k=1}^{n} |r_{jk} - \tilde{r}_{jk}|\Big)$$

with a positive constant L.

Let function $\alpha\colon D_0^* \to \mathscr{R}$, $\alpha \in E_2^l$ be a regular and parabolic solution of the equation

(2.2) $$\alpha_t(t, x) = f(t, x, \alpha(t, x), \alpha_x(t, x), \alpha_{xx}((t, x)), \quad (t, x) \in D^*.$$

and function $\beta\colon D_0 \to \mathscr{R}$, $\beta \in E_2^l$ be a regular solution of the inequality

(2.3) $$|\beta_t(t, x) - f(t, x, \beta(t, x), \beta_x(t, x), \beta_{xx}((t, x))| \leqslant N \quad \text{for } (t, x) \in D^*.$$

Suppose functions α and β satisfy the initial-boundary inequality

(2.4) $$|\beta(t, x) - \alpha(t, x)| \leqslant \varepsilon, \quad \textit{for } (t, x)\, D_0^* - D^*.$$

Then the inequality:

(2.5) $$|\alpha(t, x) - \beta(t, x)| \leqslant (\varepsilon + NL^{-1})\exp L(t-t_0) - NL^{-1} \quad \text{for } (t, x) \in \overline{D^*}.$$

holds true.

THEOREM 1. (Comparison theorem) *Let us assume that*

1°. *D has the property P and $D_0 \supset \bar{D}$, $D_0 \subset \{(t, x)\colon t \leqslant t_0 + T,\ x \in \mathscr{R}^n\}$,*

2°. *real functions $f^i(t, x, z, q, r, w)$ $(i = 1, 2, \ldots)$ satisfy Assumptions* A *and the Lipschitz condition*

(2.6) $$[f^i(t, x, z, q, r, w) - f^i(t, x, \tilde{z}, \tilde{q}, \tilde{r}, \tilde{w})]\operatorname{sgn}(z^i - \tilde{z}^i) \leqslant L\Big(\exp\Big(-K\sum_{j=1}^{l} x_j^2\Big)\|z - z\|_{l^\infty} + \sum_{j=1}^{n} |q_j - \tilde{q}_j| + \sum_{j,k=1}^{n} |r_{jk} - \tilde{r}_{jk}| + \|w - w\|_t^l\Big) \quad (i = 1, 2, \ldots),$$

3°. $u \in E_2^{l,\infty}$ *is a regular and parabolic solution of* (1.1),

4°. $v \in E_2^{l,\infty}$ *is a regular solution of the system of inequalities*

(2.7) $$|v_t^i(t,x) - f^i(t,x,v(t,x),v_x^i(t,x),v_{xx}^i(t,x),v)| \leqslant \varepsilon$$

for $(t,x) \in D$, $(i = 1,2,\ldots)$, *and a given* $\varepsilon \geqslant 0$,

5°. *u and v satisfy the initial and boundary inequalities*

(2.8) $$|u^i(t,x) - v^i(t,x)| \leqslant \varepsilon \quad \text{for } (t,x) \in D_0 - \overline{D^*} \quad (i = 1,2,\ldots).$$

Under these assumptions the following inequality

(2.9) $$||u-v||^l \leqslant \varepsilon B^q$$

holds true, where

(2.10) $$h := L^{-1}\ln\left(\frac{5}{4}\right), \quad q := \left[\frac{T}{h}\right] + 1, \quad B := (5+L^{-1})\cdot 2^{-1}$$

Proof. Let us use the Lemma and put in it at a given i

(2.11) $$D^* := D \cap \{(t,x) \colon t \leqslant t_0 + h,\ x \in \mathscr{R}^n\},$$

(2.12) $$D_0^* := D_0 \cap \{(t,x) \colon t \leqslant t_0 + h,\ x \in \mathscr{R}^n\}$$

$$\alpha(t,x) := u^i(t,x),\ \beta(t,x) := v^i(t,x) \quad \text{for } (t,x) \in D_0^* \text{ and}$$

(2.13) $$f(t,x,y,q,r := f^i(t,x,u^1(t,x),\ldots,u^{i-1}(t,x),u^{i+1}(t,x),\ldots,q,r,u)$$

$$\text{for } (t,x) \in D^*,\ y \in \mathscr{R},\ q \in \mathscr{R}^n,\ r \in \mathscr{R}^{n\times n}.$$

By the assumptions of our theorem we see that α belongs to E_2^l and is a regular and parabolic solution of (2.2). Moreover, β belongs to E_2^l and is a regular solution of the inequality

$$|\beta_t(t,x) - f(t,x,\beta(t,x),\beta_x(t,x),\beta_{xx}(t,x))| \leqslant 2L||u-v||_t^l + \varepsilon \text{ for } (t,x) \in D^*.$$

Let us notice that (2.3) holds true in D^* with

$$N := 2L||u-v||_{t_0+h}^l + \varepsilon.$$

There all assumptions of the Lemma are satisfied.

Hence it follows by (2.5) that

$$|u^i(t,x) - v^i(t,x)| \leqslant 2||u-v||_{t_0+h}^l \exp(L(t-t_0)-1) + \varepsilon[(1+L^{-1})\exp L(t-t_0) - L^{-1}]$$

for $(t,x) \in \overline{D^*}$.

From the definition of D^* and h we have

$$\exp L(t-t_0) \leqslant \tfrac{5}{4}$$

and therefore

(2.14) $$|u^i(t,x) - v^i(t,x)| \leqslant 2^{-1}||u-v||_{t_0+h}^l + 2^{-1}\varepsilon B \quad \text{for } (t,x) \in \overline{D^*}$$

where B is defined by (2.10).

Since $2^{-1}B > 1$ it follows from (2.8) that inequality (2.14) holds for $(t, x) \in D_0^*$.

Inequality (2.14) is true for every $i = 1, 2, \ldots$ This implies that

(2.15) $$||u-v||_{t_0+h}^{l} \leqslant \varepsilon B .$$

Now let us make the following substitutions in (2.11) and (2.12):

$$t_0+h \text{ for } t_0,\ D_1 := D-\overline{D^*} \text{ for } D,\ D_{01} := D_0 \cup \overline{D^*} \text{ for } D_0 .$$

From (2.14) we have

(2.16) $$|u^i(t, x)-v^i(t, x)| \leqslant \varepsilon B \quad \text{for } (t, x) \in D_{01}-D_1 .$$

Then by repetition of a similar reasoning as before for the initial and boundary inequalities (2.16) with εB instead of ε we get

$$|u^i(t, x)-v^i(t, x)| \leqslant 2^{-1}||u-v||_{t_0+2h}^{l}+2^{-1}\varepsilon B^2 \quad \text{in} \quad D_1 .$$

Hence with the view to (2.16) we conclude that

$$||u-v||_{t_0+2h}^{l} \leqslant B^2.$$

Continuing this procedure q times, where $q = \left[\frac{T}{h}\right]+1$ we obtain

(2.17) $$||u-v_{t_0+qh}^{l} \leqslant \varepsilon B^q$$

Inequality (2.17) is equivalent to our thesis because for $t_0+qh \geqslant t_0+T$ holds $||u-v||_{t_0+qh}^{l} = ||u-v||^{l}$.

From Theorem 1 one can obtain the following theorems: the uniqueness criterion and a theorem about the dependence of the solution on the right hand sides of the system and on the initial and boundary conditions.

THEOREM 2. (Uniqueness criterion) *Let us suppose assumptions* 1° *and* 2° *of Theorem* 1 *hold true. Then the system of equations* (1.1) *admits at most one regular and parabolic solution* $u \in E_2^{l,\infty}$ *satisfying the initial and boundary condition*

(2.18) $$u(t, x) = \varphi(t, x) \quad \textit{for } (t, x) \in D_0-D ,$$

where φ *is a given function*

$$\varphi = (\varphi^1, \varphi^2, \ldots): D_0-D \to l^\infty.$$

Proof. Suppose there exists a second regular solution (not necessarily parabolic) of our problem. Let us denote it by v. All assumptions of Theorem 1 are now satisfied. Because the inequality $||u-v||^{l} \leqslant \varepsilon B^q$ is true for every $\varepsilon \geqslant 0$ we conclude that $u = v$ in D_0.

THEOREM 3. (Continuous dependence of the solution on the right hand sides of the system and on the initial and boundary condition) *Let us suppose that*:

1°. *all assumptions of Theorem* 2 *hold true*,

2°. $u \in E_2^{l,\infty}$ *is the regular and parabolic solution of the problem* (1,1), (2.18), 3°. *for every* $\nu = 1, 2, \ldots$, $u \in E_2^{1,\infty}$ *is a regular solution of the system*

$$u_t^{\nu i}(t,x) = f^{\nu i}(t,x,u^\nu(t,x), u_x^{\nu i}(t,x), u_{xx}^{\nu i}(t,x), u^\nu) \quad (i = 1, 2, \ldots),\ (t,x) \in D,$$

4°.

$$\lim_{\nu\to\infty} \varepsilon_\nu = 0, \text{ where} \tag{2.19}$$

$$\begin{aligned} \eta_\nu^i := \sup\{|f^{\nu i}(t,x,u^\nu(t,x), u_x^{\nu i}(t,x), u_{xx}^{\nu i}(t,x), u^\nu) \\ -f^i(t,x,u^{\nu i}(t,x), u_x^\nu(t,x), u_{xx}^{\nu i}(t,x), u^\nu)| : (t,x) \in D\}, \\ \mu_\nu^i := \sup\{|u^{\nu i}(t,x) - \varphi^i(t,x)| : (t,x) \in D_0 - D\}, \\ \varepsilon_\nu := \sup\{\eta_\nu^i, \mu_\nu^i : i = 1, 2, \ldots\}. \end{aligned}$$

Under these assumptions

$$\lim_{\nu\to\infty} ||u^\nu - u||^l = 0. \tag{2.20}$$

Proof. For a fixed but arbitrary index ν all assumptions of Theorem 1 are satisfied with $v = u^\nu$ and $\varepsilon = \varepsilon_\nu$.

Inequality (2.9) yields

$$||u^\mu - u||^l \leqslant \varepsilon_\nu B^q \quad (\nu = 1, 2, \ldots).$$

The latter inequalities together with (2.19) imply (2.20).

Remarks. Theorems 1, 2 and 3 remain true if inequality (2.6) is replaced by

$$[f^i(t,x,z,q,r,w) - f^i(t,x,\tilde{z},\tilde{q},\tilde{r},\tilde{w})]\operatorname{sgn}(z^i - \tilde{z}^i) \tag{2.21}$$

$$\leqslant L(|z^i - \tilde{z}^i| + \sum_{j=1}^n |q_j - \tilde{q}_j| + \sum_{j=k=1}^n |r_{jk} - \tilde{r}_{jk}| + ||w - w||)_t^l \ (i = 1, 2, \ldots)$$

and if constants h, q and B in the thesis of Theorem 1 are selected to be:
$h := L^{-1}\ln\left(\frac{3}{2}\right)$, $q := \left[\frac{T}{h}\right] + 1$, $B := 3 + L^{-1}$.

3. Theorems for a strongly coupled infinite system of parabolic equations.

THEOREM 4. (Comparison theorem) *Let* D *have the property* P *and let us put* $D_0 = \bar{D}$. *Suppose that* D *is of the particular form* $D := D_{n-1} \times \mathcal{R}$, *where* $D_{n-1} \subset \{(t, x_1, \ldots, x_{n-1}) \in \mathcal{R}^n\}$. *Let real functions* $\tilde{f}^i(t,x,z,s,q,r)$ $(i = 1, 2, \ldots)$ *be defined for*

$$(t,x) \in D,\ z \in l^\infty,\ s \in l^\infty,\ q \in \mathcal{R}^n,\ r \in \mathcal{R}^{n\times n},$$

and satisfy the Lipschitz condition

(3.1) $$[\tilde{f}^i(t,x,z,s,q,r)-\tilde{f}^i(t,x,\tilde{z},\tilde{s},\tilde{q},\tilde{r})]\operatorname{sgn}(z^i-\tilde{z}^i)$$
$$\leqslant \tilde{L}\big(\exp(-K\sum_{j=1}^{n-}x_j^2)(\|z-\tilde{z}\|_{l^\infty}+\|s-\tilde{s}\|_{l^\infty})+\sum_{i=1}^{n}q_i-\tilde{q}_i+\sum_{j=1}|r_{ij}-\tilde{r}_{ij}|\big)\ (i=1,2,\dots)$$

with a positive constant L.

Let furthermore a function $u\in Y$, *where* $Y:=E_2^{n-1,\infty}\cap\mathscr{L}(\kappa,x_n\colon D)$, $(\kappa>0)$ be a regular and parabolic solution of the strongly coupled system

(3.2) $$u_t^i(t,x)=\tilde{f}^i(t,x,u(t,x),u_{x_n}(t,x),u_x^i(t,x),u_{xx}^i(t,x))\ (i=1,2,\dots)\ \textit{for}\ (t,x)\in D,$$

where $u_{x_n}=(u_{x_n}^1,u_{x_n}^2,\dots)$.

Let finally $v\in Y$ *be a regular solution to the system of inequalities*

(3.3) $$|v_t^i(t,x)-\tilde{f}^i(t,x,v(t,x),v_{x_n}(t,x),v_x^i(t,x),v_{xx}^i(t,x))|\leqslant\varepsilon\ (i=1,2,\dots),\ (t,x)\in D$$

for a given $\varepsilon\geqslant 0$ *and let v satisfy the initial and boundary inequalities* (2.8).

Under these assumptions the inequality

(3.4) $$\|u-v\|^{n-1}\leqslant\varepsilon B^q$$

holds true, where

$$h:=L^{-1}\ln(5/4),\quad q:=[T/h]+1,\quad B:=(5+L^{-1})2^{-1}$$

with

(3.5) $$L:=\max\{\tilde{L},\varkappa\tilde{L}\}.$$

Proof. A solution $u\in E_2^{n-1,\infty}\cap\mathscr{L}(\varkappa,x_n\colon D)$ of system (3.2) can be treated as a solution of the weakly coupled system of parabolic differential-functional equations

(3.6) $$u_t^i(t,x)=f^i(t,x,u(t,x),u_x^i(t,x),u_{xx}^i(t,x),u),$$

where

(3.7) $$f^i(t,x,z,q,r,w)=\tilde{f}^i(t,x,z,w_{x_n}(t,x),q,r)\quad(i=1,2,\dots)$$

For u, $v\in Y$ and functions f^i so defined we shall show that all assumptions of Theorem 1 are satisfied.

Let w, $\tilde{w}\in Y$. From Bernstein's theorem we have

(3.8) $$|w_{x_n}^i(t,x)-\tilde{w}_{x_n}^i(t,x)|\leqslant\varkappa\sup\{|w^i(t,x)-\tilde{w}^i(t,x)|\colon x_n\in\mathscr{R}\}\quad(i=1,2,\dots).$$

Applying the definition of the norm $\|\ \|_t^{n-1}$ we obtain

(3.9) $$\sup\{|w^i(t,x)-\tilde{w}^i(t,x)|\colon x_n\in\mathscr{R}\}\leqslant\|w-w_t^{n-1}\|\exp(K\sum_{j=1}^{n-1}x_j^2).$$

From (3.8) and (3.9) we conclude that

$$|w_{x_n}^i(t,x)-\tilde{w}_{x_n}^i(t,x)|\leqslant\varkappa\|w-\tilde{w}\|_t^{n-1}\exp(K\sum_{j=1}^{n-1}x_j^2)\quad(i=1,2,\dots)$$

and hence

$$\|w_{x_n}(t,x)-\tilde{w}_{x_n}(t,x)\|_{l\infty} \leqslant \varkappa \|w-\tilde{w}\|_t^{n-1} \exp\Big(K\sum_{j=1}^{n-1} x_j^2\Big). \tag{3.10}$$

Using (3.1), (3.7) and (3.10) we get

$$[f^i(t,x,z,q,r,w)-f^i(t,x,\tilde{z},\tilde{q},\tilde{r},\tilde{w})]\operatorname{sgn}(z^i-\tilde{z}^i) \leqslant \tilde{L}\exp\Big(-K\sum_{j=1}^{n-1} x_j^2\Big)\|z-\tilde{z}\|_{l\infty} + \varkappa\tilde{L}\|w-\tilde{w}\|_t^{n-1} + \tilde{L}\Big(\sum_{j=1}^{n}|q_j-\tilde{q}_j| + \sum_{i,j=1}^{n}|r_{ij}-\tilde{r}_{ij}|\Big) \quad (i=1,2,\ldots). \tag{3.11}$$

Then, by (3.11), we see that the Lipschitz condition (2.6) is fulfilled with L defined by (3.5). We know that $u, v \in Y$ and, moreover, by (3.1), (3.3), that u is a regular and parabolic solution of system (1.1) with f^i defined by (3.6) and that v is a regular solution of (2.7). The functions u and v satisfy also initial and boundary inequalities (2.8).

Now, using Theorem 1, we can conclude that our thesis (3.4) holds true.

For the case of solutions of the system (3.2) belonging to the class $E_2^{n-1,\infty} \cap \mathscr{L}(\varkappa, x_n\colon D)$ one can easily prove by means of Theorem 4 the uniqueness criterion and the theorem stating the continuous dependence of the solution of the right hand sides of the system and on the initial and boundary function. The two theorems are obtained from Theorems 2 and 3 by simple reformulation.

Remarks. Theorems formulated in this chapter remain true if inequality (3.1) is replaced by the inequality

$$[f^i(t,x,z,s,q,r)-f^i(t,x,\tilde{z},\tilde{s},\tilde{q},\tilde{r})]\operatorname{sgn}(z^i-\tilde{z}^i)$$
$$\leqslant L\Big(|z^i-\tilde{z}^i| + \|s-\tilde{s}\|_{l\infty}\exp\Big(-K\sum_{j=1}^{n-1}x_j^2\Big) + \sum_{j=1}^{n}|q_j-\tilde{q}_j| + \sum_{i,j=1}^{n}|r_{ij}-\tilde{r}_{ij}|\Big) \quad (i=1,2,\ldots)$$

and if constant h, q, and B in the thesis of Theorem 4 are chosen to be

$$h := L^{-1}\ln(3/2),\ q := [T/h]+1,\ B := 3+L^{-1}.$$

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