

## On Series of Homogeneous Polynomials Noncontinuable Beyond Their Domain of Convergence

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**Abstract.** Let  $\Omega$  be a balanced domain of holomorphy. Then functions which are holomorphic in  $\Omega$  and noncontinuable beyond  $\Omega$  form a large set in  $\mathcal{O}_\Omega$  equipped with a suitable topology.

**0. Introduction.** Let  $f$  be an analytic function of  $n$  complex variables defined by a series of homogeneous polynomials

$$f(z) = \sum_{v=0}^{\infty} f_v(z), \quad (\deg f_v = v) \quad (0.1)$$

that converges in a neighbourhood of zero in the space  $\mathbb{C}^n$ .

Let  $\Omega$  denote the domain of convergence of series (0.1). Since  $\Omega$  is the domain of holomorphy, there exists at least one function  $f$  holomorphic in  $\Omega$  and noncontinuable beyond  $\Omega$ . Let us consider the set  $\mathcal{O}_\Omega$  of all functions analytic in  $\Omega$  and a subset of  $\mathcal{O}_\Omega$  consisting of those functions which are noncontinuable beyond  $\Omega$ . The question arises how large this subset is. L. Bieberbach in his book [2] presented the history of this problem and collected many theorems concerning it for  $n = 1$ .

The aim of this paper is to generalize three of the results inserted in [2]. Two of them, due to Hausdorff and Polya, say that having set up a suitable topology in  $\mathcal{O}_\Delta$ , the subset under consideration turns out to be dense and open in  $\mathcal{O}_\Delta$ ,  $\Delta$  being the unit disc in  $\mathbb{C}$ . The third one, due to Ryll-Nardzewski and Steinhaus, says that in a Banach space consisting of functions holomorphic in the unit disc the functions noncontinuable beyond the disc form a set of second category.

### 1. Denotations and definitions.

$$\Delta := \{\lambda \in \mathbb{C} : |\lambda| < 1\}, \quad B(z_0, r) := \{z \in \mathbb{C}^n : \|z - z_0\| < r\}, \quad z_0 \in \mathbb{C}^n, \quad r > 0,$$

$\|z\|$  being the Euclidean norm in  $\mathbb{C}^n$ .

For a function  $f: \mathbb{C}^n \supset S \rightarrow \mathbb{C}$  we write

$$\|f\|_S = \sup \{|f(z)| : z \in S\}.$$

$P^v(\mathbf{C}^n, \mathbf{C})$  denotes the set of all homogeneous polynomials from  $\mathbf{C}^n$  to  $\mathbf{C}$  of degree  $v$  ( $v = 0, 1, 2, \dots$ ).

If  $f: \Omega \rightarrow \mathbf{C}$  is analytic in  $\Omega$ , then  $T_a f$  denotes the Taylor series of the function  $f$  centred in  $a \in \Omega$ ,  $\rho(T_a f)$  denotes the radius of convergence of this series.

Let  $f$  be holomorphic in a domain  $\Omega$ . We say that  $f$  continues through a point  $z_0 \in \partial\Omega$  if there exists a point  $a \in \Omega$  such that  $\rho(T_a f) > \|z_0 - a\|$ . A holomorphic function  $f: \Omega \rightarrow \mathbf{C}$  which continues through a point of  $\partial\Omega$  is called continuable beyond  $\Omega$ .

**2. Generalization of the theorems of Hausdorff and Polya.** Let  $\Omega \subset \mathbf{C}^n$  be a bounded balanced domain of holomorphy. Let  $\mathcal{O}_\Omega = \{f: \Omega \rightarrow \mathbf{C}: f \text{ is analytic}\}$ . For every  $f \in \mathcal{O}_\Omega$  we have the only representation

$$f(z) = \sum_{v=0}^{\infty} f_v(z), \quad z \in \Omega, \quad (2.1)$$

where  $f_v \in P^v(\mathbf{C}^n, \mathbf{C})$  for  $v = 0, 1, 2, \dots$

For  $f \in \mathcal{O}_\Omega$  we put

$$u_f(z) = \limsup_{v \rightarrow \infty} \sqrt[v]{|f_v(z)|}$$

and

$$u_f^*(z) = \limsup_{\zeta \rightarrow z} u_f(\zeta).$$

Then  $\Omega \subset \{z \in \mathbf{C}^n: u_f^*(z) < 1\}$ .

Following Hausdorff we introduce in  $\mathcal{O}_\Omega$  a topology defined by a fundamental system of neighbourhoods. Let  $\varepsilon = (\varepsilon_v)_{v=0}^{\infty}$  be a sequence of positive real numbers such that  $\lim_{v \rightarrow \infty} \sqrt[v]{\varepsilon_v} = 1$  and let  $f \in \mathcal{O}_\Omega$ . By an  $\varepsilon$ -neighbourhood of  $f$  we mean the set

$$V_{\Omega, \varepsilon}(f) = \left\{ g \in \mathcal{O}_\Omega: \sup_{z \in \bar{\Omega}} \left( \limsup_{v \rightarrow \infty} \frac{|f_v(z) - g_v(z)|}{\varepsilon_v} \right) < 1 \right\}.$$

Now we define an equivalence relation in  $\mathcal{O}_\Omega$  as follows: let two functions  $f, g \in \mathcal{O}_\Omega$  be in relation ( $f \sim g$ ) if and only if  $f - g \in \mathcal{O}_{\bar{\Omega}}$ ; where  $\mathcal{O}_{\bar{\Omega}} = \text{ind} \lim_{V \supset \bar{\Omega}, V \text{ open}} \mathcal{O}_V$ .

**2.1 PROPOSITION.** Let  $f, g \in \mathcal{O}_\Omega$ . Then  $f \sim g$  if and only if  $u_{f-g}^*(z) < 1$  for every  $z \in \partial\Omega$ .

*Proof.* Let  $f, g \in \mathcal{O}_\Omega$  and  $f - g \in \mathcal{O}_{\bar{\Omega}}$ . Then there exists a number  $r > 0$  such that  $f - g \in \mathcal{O}_{\bar{\Omega}^r}$ , where

$$\bar{\Omega}^r = \{z \in \mathbf{C}^n: \text{dist}(z, \bar{\Omega}) < r\},$$

$\text{dist}(z, \bar{\Omega})$  denotes the distance between the point  $z$  and the set  $\bar{\Omega}$  in Euclidean norm in  $\mathbf{C}^n$ . Since  $\bar{\Omega}^r$  is a balanced neighbourhood of zero in  $\mathbf{C}^n$ , the inequality  $u_{f-g}^*(z) < 1$  holds for  $z \in \bar{\Omega}^r$  and, in particular, for  $z \in \partial\Omega$ .

Suppose now that  $u_{f-g}^*(z) < 1$  for every  $z \in \partial\Omega$ . Then  $\partial\Omega$  so  $\bar{\Omega}$  is contained in the domain of convergence of series  $\sum_{v=0}^{\infty} (f_v - g_v)$ . Therefore  $f \sim g$ .

