

## On Series of Homogeneous Polynomials Noncontinuable Beyond Their Domain of Convergence

by Małgorzata DOWNAROWICZ

**Abstract.** Let  $\Omega$  be a balanced domain of holomorphy. Then functions which are holomorphic in  $\Omega$  and noncontinuable beyond  $\Omega$  form a large set in  $\mathcal{O}_\Omega$  equipped with a suitable topology.

**0. Introduction.** Let  $f$  be an analytic function of  $n$  complex variables defined by a series of homogeneous polynomials

$$f(z) = \sum_{v=0}^{\infty} f_v(z), \quad (\deg f_v = v) \quad (0.1)$$

that converges in a neighbourhood of zero in the space  $\mathbb{C}^n$ .

Let  $\Omega$  denote the domain of convergence of series (0.1). Since  $\Omega$  is the domain of holomorphy, there exists at least one function  $f$  holomorphic in  $\Omega$  and noncontinuable beyond  $\Omega$ . Let us consider the set  $\mathcal{O}_\Omega$  of all functions analytic in  $\Omega$  and a subset of  $\mathcal{O}_\Omega$  consisting of those functions which are noncontinuable beyond  $\Omega$ . The question arises how large this subset is. L. Bieberbach in his book [2] presented the history of this problem and collected many theorems concerning it for  $n = 1$ .

The aim of this paper is to generalize three of the results inserted in [2]. Two of them, due to Hausdorff and Polya, say that having set up a suitable topology in  $\mathcal{O}_\Delta$ , the subset under consideration turns out to be dense and open in  $\mathcal{O}_\Delta$ ,  $\Delta$  being the unit disc in  $\mathbb{C}$ . The third one, due to Ryll-Nardzewski and Steinhaus, says that in a Banach space consisting of functions holomorphic in the unit disc the functions noncontinuable beyond the disc form a set of second category.

### 1. Denotations and definitions.

$$\Delta := \{\lambda \in \mathbb{C} : |\lambda| < 1\}, \quad B(z_0, r) := \{z \in \mathbb{C}^n : \|z - z_0\| < r\}, \quad z_0 \in \mathbb{C}^n, \quad r > 0,$$

$\|z\|$  being the Euclidean norm in  $\mathbb{C}^n$ .

For a function  $f: \mathbb{C}^n \supset S \rightarrow \mathbb{C}$  we write

$$\|f\|_S = \sup \{|f(z)| : z \in S\}.$$

$P^v(\mathbf{C}^n, \mathbf{C})$  denotes the set of all homogeneous polynomials from  $\mathbf{C}^n$  to  $\mathbf{C}$  of degree  $v$  ( $v = 0, 1, 2, \dots$ ).

If  $f: \Omega \rightarrow \mathbf{C}$  is analytic in  $\Omega$ , then  $T_a f$  denotes the Taylor series of the function  $f$  centred in  $a \in \Omega$ ,  $\rho(T_a f)$  denotes the radius of convergence of this series.

Let  $f$  be holomorphic in a domain  $\Omega$ . We say that  $f$  continues through a point  $z_0 \in \partial\Omega$  if there exists a point  $a \in \Omega$  such that  $\rho(T_a f) > \|z_0 - a\|$ . A holomorphic function  $f: \Omega \rightarrow \mathbf{C}$  which continues through a point of  $\partial\Omega$  is called continuable beyond  $\Omega$ .

**2. Generalization of the theorems of Hausdorff and Polya.** Let  $\Omega \subset \mathbf{C}^n$  be a bounded balanced domain of holomorphy. Let  $\mathcal{O}_\Omega = \{f: \Omega \rightarrow \mathbf{C}: f \text{ is analytic}\}$ . For every  $f \in \mathcal{O}_\Omega$  we have the only representation

$$f(z) = \sum_{v=0}^{\infty} f_v(z), \quad z \in \Omega, \quad (2.1)$$

where  $f_v \in P^v(\mathbf{C}^n, \mathbf{C})$  for  $v = 0, 1, 2, \dots$

For  $f \in \mathcal{O}_\Omega$  we put

$$u_f(z) = \limsup_{v \rightarrow \infty} \sqrt[v]{|f_v(z)|}$$

and

$$u_f^*(z) = \limsup_{\zeta \rightarrow z} u_f(\zeta).$$

Then  $\Omega \subset \{z \in \mathbf{C}^n: u_f^*(z) < 1\}$ .

Following Hausdorff we introduce in  $\mathcal{O}_\Omega$  a topology defined by a fundamental system of neighbourhoods. Let  $\varepsilon = (\varepsilon_v)_{v=0}^{\infty}$  be a sequence of positive real numbers such that  $\lim_{v \rightarrow \infty} \sqrt[v]{\varepsilon_v} = 1$  and let  $f \in \mathcal{O}_\Omega$ . By an  $\varepsilon$ -neighbourhood of  $f$  we mean the set

$$V_{\Omega, \varepsilon}(f) = \left\{ g \in \mathcal{O}_\Omega: \sup_{z \in \bar{\Omega}} \left( \limsup_{v \rightarrow \infty} \frac{|f_v(z) - g_v(z)|}{\varepsilon_v} \right) < 1 \right\}.$$

Now we define an equivalence relation in  $\mathcal{O}_\Omega$  as follows: let two functions  $f, g \in \mathcal{O}_\Omega$  be in relation ( $f \sim g$ ) if and only if  $f - g \in \mathcal{O}_{\bar{\Omega}}$ ; where  $\mathcal{O}_{\bar{\Omega}} = \text{ind} \lim_{V \supset \bar{\Omega}, V \text{ open}} \mathcal{O}_V$ .

**2.1 PROPOSITION.** Let  $f, g \in \mathcal{O}_\Omega$ . Then  $f \sim g$  if and only if  $u_{f-g}^*(z) < 1$  for every  $z \in \partial\Omega$ .

*Proof.* Let  $f, g \in \mathcal{O}_\Omega$  and  $f - g \in \mathcal{O}_{\bar{\Omega}}$ . Then there exists a number  $r > 0$  such that  $f - g \in \mathcal{O}_{\bar{\Omega}^r}$ , where

$$\bar{\Omega}^r = \{z \in \mathbf{C}^n: \text{dist}(z, \bar{\Omega}) < r\},$$

$\text{dist}(z, \bar{\Omega})$  denotes the distance between the point  $z$  and the set  $\bar{\Omega}$  in Euclidean norm in  $\mathbf{C}^n$ . Since  $\bar{\Omega}^r$  is a balanced neighbourhood of zero in  $\mathbf{C}^n$ , the inequality  $u_{f-g}^*(z) < 1$  holds for  $z \in \bar{\Omega}^r$  and, in particular, for  $z \in \partial\Omega$ .

Suppose now that  $u_{f-g}^*(z) < 1$  for every  $z \in \partial\Omega$ . Then  $\partial\Omega$  so  $\bar{\Omega}$  is contained in the domain of convergence of series  $\sum_{v=0}^{\infty} (f_v - g_v)$ . Therefore  $f \sim g$ .

**2.2.** Now let us consider the topological space  $\mathcal{X} = \mathcal{O}_{\Omega}/\sim$  with the quotient topology. Let  $\mathcal{N} = \{[f] \in \mathcal{X} : f \text{ is noncontinuable beyond } \Omega\}$ .

**2.3. THEOREM** (for  $n = 1$  see Theorem 4.2.1, in [2]). *Let  $\Omega$  be a bounded balanced domain of holomorphy in  $\mathbb{C}^n$ . Suppose that there exists a family  $P$  of homogeneous polynomials such that*

$$\Omega = \text{int} \{z \in \mathbb{C}^n : |p(z)| < 1, p \in P\}.$$

*Then the set  $\mathcal{N}$  is dense in  $\mathcal{X}$ .*

**Proof.** 1°. Choose an arbitrary sequence  $\varepsilon = (\varepsilon_v)_{v=0}^{\infty}$  of positive real numbers such that  $\sqrt[v]{\varepsilon_v} \rightarrow 1$ , with  $v \rightarrow \infty$ . We shall find a series of homogeneous polynomials  $\sum_{v=0}^{\infty} g_v$ ,  $g_v \in P^v(\mathbb{C}^n, \mathbb{C})$ , convergent in  $\Omega$  to a function  $g$  noncontinuable beyond  $\Omega$  and such that  $\|g_v\|_{\Omega} \leq \varepsilon_v$ ,  $v = 0, 1, 2, \dots$

By hypothesis there exists a countable set  $A = \{a_1, a_2, \dots\}$  contained in  $\mathbb{C}^n \setminus \Omega$  satisfying the two following conditions:

- (i)  $\partial\Omega \subset \bar{A}$ ,
- (ii) for every  $z \in A$  there exists a homogeneous polynomial  $p$ , not identically equal to 1 such that  $|p(z)| = 1$  and  $\|p\|_{\Omega} \leq 1$ .

Let  $(b_s)_{s \in \mathbb{N}}$  be a sequence of elements of  $A$  such that each element of  $A$  is repeated in  $(b_s)_{s \in \mathbb{N}}$  infinitely many times. By virtue of (ii) for every  $s \in \mathbb{N}$  we can choose a homogeneous polynomial  $p_s \neq 1$  such that  $\|p_s\|_{\Omega} \leq 1$  and  $|p_s(b_s)| = 1$ . Given a sequence of positive integers  $(m_s)_{s \in \mathbb{N}}$  such that  $m_s \deg p_s > 2m_{s-1} \deg p_{s-1}$ ,  $s \geq 2$ , put

$$\begin{aligned} f_1 &= p_1, \\ f_s &= p_s^{m_s}, \quad s \geq 2 \end{aligned}$$

and write  $v_s := \deg f_s = m_s \deg p_s$ . For an arbitrary integer  $v \geq 0$ , we define

$$g_v := \begin{cases} 0, & \text{if } v \notin \{v_s : s \in \mathbb{N}\} \\ \varepsilon_{v_s} f_s, & \text{if } v = v_s \text{ for some } s \in \mathbb{N}. \end{cases}$$

Observe that for every  $z \in \partial\Omega$

$$\limsup_{v \rightarrow \infty} \sqrt[v]{|g_v(z)|} \leq \lim_{v \rightarrow \infty} \sqrt[v]{\varepsilon_v} = 1.$$

Hence the series  $\sum_{v=0}^{\infty} g_v$  converges in  $\Omega$ . Moreover,

$$\limsup_{v \rightarrow \infty} \sqrt[v]{|g_v(a)|} \geq 1, \quad a \in A.$$

So  $\Omega$  is the domain of convergence of  $\sum_{v=0}^{\infty} g_v$ . Let

$$g(z) = \sum_{v=0}^{\infty} g_v(z), \quad z \in \Omega.$$

Since the series under consideration is a lacunary one, the function  $g$  cannot be continued beyond  $\Omega$  ([13]).

2°. Let  $f \in \mathcal{O}_\Omega$ . Given  $g$ , constructed in 1°, let us consider the family of functions

$$f_\alpha = \sum_{v=0}^{\infty} (f_v + \alpha g_v), \quad \alpha \in (0, 1).$$

Note that for any  $\alpha \in (0, 1)$ ,  $f_\alpha \notin [f]$ . Indeed,

$$\limsup_{v \rightarrow \infty} \sqrt[v]{|g_v(a)|} \geq 1, \quad a \in A.$$

Hence, in view of (i),

$$u_{f-f_\alpha}^*(z) = \limsup_{\zeta \rightarrow z} (\limsup_{v \rightarrow \infty} \sqrt[v]{\alpha} \sqrt[v]{|g_v(\zeta)|}) \geq 1, \quad z \in \partial\Omega.$$

So, by Proposition 2.1,  $f_\alpha \notin [f]$ . As a consequence of the construction of  $g$  we obtain

$$f_\alpha \in V_{\Omega, \varepsilon}(f), \quad \alpha \in (0, 1).$$

We claim that at least one of the functions  $f_\alpha$  cannot be continued beyond  $\Omega$ . Otherwise for every  $\alpha \in (0, 1)$  there would exist a point  $z \in \Omega$  and a positive integer  $k$  such that  $\varrho(T_z f_\alpha) \geq \text{dist}(z, \partial\Omega) + 1/k$ . Let  $D$  be a countable subset of  $\Omega$ , dense in  $\Omega$ . To every  $\alpha \in (0, 1)$  there corresponds a pair  $(z, k) \in D \times N$  such that  $\varrho(T_z f_\alpha) \geq \text{dist}(z, \partial\Omega) + 1/k$ . In view of the countability of  $D \times N$  and the uncountability of  $(0, 1)$  we can find two different numbers  $\alpha_1, \alpha_2 \in (0, 1)$  and a pair  $(z, k) \in D \times N$  such that

$$\varrho(T_z f_{\alpha_i}) \geq \text{dist}(z, \partial\Omega) + 1/k \quad (i = 1, 2).$$

Therefore  $\varrho(T_z(f_{\alpha_1} - f_{\alpha_2})) \geq \text{dist}(z, \partial\Omega) + 1/k$ .

$$\text{But } (f_{\alpha_1} - f_{\alpha_2})(z) = (\alpha_1 - \alpha_2) \sum_{v=0}^{\infty} g_v(z) = (\alpha_1 - \alpha_2)g(z), \quad z \in \Omega.$$

This leads to a contradiction since  $g$  is noncontinuable beyond  $\Omega$ . The proof is ended.

A class of domains which satisfy the hypothesis of Theorem 2.3 is given by the following

**2.4. PROPOSITION.** *Let  $\Omega$  be a bounded balanced domain of holomorphy in  $\mathbb{C}^n$ . Assume that  $\partial\Omega$  does not contain a ring (i.e. the intersection of  $\partial\Omega$  with any complex vector line is a circle). Then there exists a family  $P$  of homogeneous polynomials from  $\mathbb{C}^n$  to  $\mathbb{C}$  such that*

$$\Omega = \text{int} \{z \in \mathbb{C}^n : |p(z)| < 1, p \in P\}.$$

**Proof.** We define  $P := \{p: \mathbb{C}^n \rightarrow \mathbb{C} : p \text{ is a homogeneous polynomial, } \|p\|_\Omega = 1, \text{deg } p \geq 1\}$ . Then

$$\Omega \subset \Omega' := \text{int} \bigcap_{p \in P} \{z \in \mathbb{C}^n : |p(z)| < 1\}.$$

We shall prove that  $\Omega' \subset \Omega$ . With this aim in view fix  $z_0 \in \Omega'$  and take a number  $r > 1$  such that  $rz_0 \in \Omega'$ . Then for every  $p \in P$   $|p(rz_0)| < 1$ . Therefore the point  $rz_0$  belongs to the hull of  $\bar{\Omega}$  convex with respect to homogeneous polynomials. By hypothesis  $\bar{\Omega}$  is a compact subset of  $r\Omega$ . But  $r\Omega$ , as a balanced domain of holomorphy, is convex with respect to homogeneous polynomials. Hence  $rz_0 \in r\Omega$ , so  $z_0 \in \Omega$ . The proof is completed.

2.5. Remark. Note that there exist balanced domains of holomorphy of the form

$$\Omega = \text{int} \bigcap_{p \in P} \{z \in \mathbb{C}^n : |p(z)| < 1\},$$

the boundaries of  $\Omega$  contain rings,  $P$  being a family of homogeneous polynomials (see Theorems 3.1 and 4.1 in [11]).

2.6. LEMMA. Let  $\Omega$  be a bounded balanced domain in  $\mathbb{C}^n$  such that  $\partial\Omega$  does not contain a ring. Assume that  $f \in \mathcal{O}_\Omega$  is noncontinuable beyond  $\Omega$ . Put  $\mathcal{A} = \{z \in \partial\Omega : f_z \text{ continues analytically through } 1\}$ , where  $f_z : \Delta \in \lambda \rightarrow f(\lambda z) \in \mathbb{C}$ ,  $z \in \partial\Omega$ . Then  $\mathcal{A} = \mathcal{C} \cap \partial\Omega$ ,  $\mathcal{C}$  being a pluripolar cone in  $\mathbb{C}^n$ .

Proof. By hypothesis  $\Omega$  is the Mittag-Leffler star of the function  $f$  (for definition see [5]). Let

$$f(z) = \sum_{v=0}^{\infty} f_v(z), \quad z \in \Omega; \quad f_v \in P^v(\mathbb{C}^n, \mathbb{C}), \quad v = 0, 1, 2, \dots$$

For every  $k \in \mathbb{N}$  let us consider the  $k$ -th function associated with  $f$  given by the formula

$$F_k(z) = \sum_{v=0}^{\infty} \frac{f_v(z)}{\Gamma(1+v/k)}, \quad z \in \mathbb{C}^n$$

and its regularized radial indicator

$$H_k^*(z) = \limsup_{\zeta \rightarrow z} H_k(\zeta),$$

where

$$H_k(\zeta) = \limsup_{t \rightarrow \infty} t^{-k} \ln |F_k(t\zeta)|, \quad \zeta \in \mathbb{C}^n.$$

Note that for  $n = 1$  the function  $H_k$  is continuous ([6]), so in that case  $H_k = H_k^*$ ,  $k = 1, 2, \dots$

We claim that  $\mathcal{A} = \bigcup_{k=1}^{\infty} A_k$ , where  $A_k = \{z \in \partial\Omega : H_k(z) < H_k^*(z)\}$ . Indeed, given any  $z \in \partial\Omega$ , we have:  $f_z$  continues analytically through 1 if and only if there exists  $k \in \mathbb{N}$  such that  $H_k(z) < 1$ . This fact is a consequence of Theorem 4 in [5] for  $n = 1$  (since  $H_k(z)$  is the value at 1 of the indicator of  $k$ -th function associated with  $f_z$ ). Furthermore, by the assumptions on  $\Omega$  and by Theorem 4 in [5] we obtain  $\Omega = \{z \in \mathbb{C}^n : H_k^*(z) < 1\}$ ,  $k = 1, 2, \dots$  and  $H_k^*(z) = 1$ ,  $z \in \partial\Omega$ ,  $k = 1, 2, \dots$ . Hence, for  $z \in \partial\Omega$ ,  $f_z$  continues analytically through 1 if and only if there exists  $k \in \mathbb{N}$  such that  $H_k(z) < H_k^*(z)$ .

Let  $\mathcal{C} = \bigcup_{t \in (0, \infty)} t\mathcal{A}$ . Then  $\mathcal{C} = \bigcup_{k=1}^{\infty} \{z \in \mathbb{C}^n : H_k(z) < H_k^*(z)\}$ , since the functions  $H_k$  and  $H_k^*$  are positively homogeneous (of order  $k$ ). Hence  $\mathcal{C}$  is negligible (see [9]) and thus pluripolar ([1]).

**2.7. PROPOSITION.** *Let  $\Omega$  and  $f \in \mathcal{O}_\Omega$  satisfy the assumptions of Lemma 2.6. Put  $\mathcal{E} := \{z \in \partial\Omega : f_z \text{ continues analytically beyond } \Delta\}$ . Then the cone  $\bigcup_{t \in (0, \infty)} t\mathcal{E}$  is pluripolar.*

*Proof.* Let  $\{\theta_1, \theta_2, \dots\} = Q \cap [0, 2\pi)$  where  $Q$  denotes the set of rational numbers. Given any  $j \in N$  we write

$$(1) f_j(z) = f(e^{i\theta_j}z), \quad z \in \Omega,$$

$$(2) E_j = \{z \in \partial\Omega : f_{j,z} \text{ continues analytically through } 1\},$$

where  $f_{j,z} : \Delta \ni \lambda \rightarrow f_j(\lambda z) \in \mathbb{C}$ .

Then  $\mathcal{E} = \bigcup_{j=1}^{\infty} E_j$ . By Lemma 2.6 the set  $\mathcal{C}_j = \bigcup_{t \in (0, \infty)} tE_j$  is pluripolar for every  $j \in N$ .

Hence  $\bigcup_{t \in (0, \infty)} t\mathcal{E} = \bigcup_{j=1}^{\infty} \mathcal{C}_j$  is pluripolar as a countable union of pluripolar sets.

**2.8. Remark.** Since the set  $\mathcal{E}$  is of the type  $F_\sigma$  it is of the first category in  $\partial\Omega$ . In the case when  $\Omega$  is a ball in  $\mathbb{C}^n$  this fact was proved (in another way) by Cima and Globevnik ([4]).

**2.9. COROLLARY.** *Let  $\Omega$  satisfy the assumptions of Lemma 2.6 and let  $f \in \mathcal{O}_\Omega$ . Then  $f$  is noncontinuable beyond  $\Omega$  if and only if there exists a subset  $Z$  of  $\partial\Omega$  dense in  $\partial\Omega$  and such that  $f_z$  is noncontinuable beyond  $\Delta$  for every  $z \in Z$ .*

**2.10. THEOREM** (for  $n = 1$  see Theorem 4.2.2 in [2]). *Let  $\Omega$  be a bounded balanced domain of holomorphy in  $\mathbb{C}^n$  such that  $\partial\Omega$  does not contain a ring. Then  $\mathcal{N}$  is open in  $\mathcal{X}$ .*

*Proof.* Let  $f \in \mathcal{O}_\Omega$  be noncontinuable beyond  $\Omega$ . We shall find a neighbourhood of  $f$  consisting of functions which are not continuable beyond  $\Omega$ .

Let  $Z = \{z_1, z_2, \dots\}$  be a countable subset of  $\partial\Omega$ , dense in  $\partial\Omega$  and such that  $f_z$  is noncontinuable beyond  $\Delta$  for every  $z \in Z$ . By Theorem 4.2.2 in [2] for each  $k \in N$  we can choose a sequence  $\varepsilon^{(k)} = (\varepsilon_v^{(k)})_{v=0}^{\infty}$  of positive real numbers satisfying the following conditions

$$(1) \lim_{v \rightarrow \infty} \sqrt[v]{\varepsilon_v^{(k)}} = 1,$$

$$(2) \text{ for every } \varphi \in V_{\Delta, \varepsilon^{(k)}}(f_{z_k}) \text{ } \varphi \text{ is noncontinuable beyond } \Delta.$$

Now let us take a strictly increasing sequence  $(\mu_k)_{k=1}^{\infty}$  of positive integers such that for any  $k \in N$

$$\sqrt[v]{\varepsilon_v^{(j)}} \geq 1 - \frac{1}{k} \quad \text{if } j \in \{1, 2, \dots, k\}, \quad v \geq \mu_k.$$

Finally we define a new sequence  $\varepsilon = (\varepsilon_v)_{v=0}^{\infty}$  as follows

$$\varepsilon_0 = \varepsilon_1 = \dots = \varepsilon_{\mu_1-1} = 1,$$

$$\varepsilon_v = \min\{\varepsilon_v^{(1)}, \varepsilon_v^{(2)}, \dots, \varepsilon_v^{(k)}\}, \quad \mu_k \leq v < \mu_{k+1}, \quad k \in N.$$

Then  $\lim_{v \rightarrow \infty} \sqrt[v]{\varepsilon_v} = 1$  and for each  $k \in N$

$$\varepsilon_v \leq \varepsilon_v^{(k)}, \quad v \geq \mu_k. \tag{2.2}$$

We claim that any function  $g \in V_{\Omega, \varepsilon}(f)$  is noncontinuable beyond  $\Omega$ . Indeed, let  $g \in V_{\Omega, \varepsilon}(f)$ . Then for every  $k \in \mathbb{N}$   $g_{z_k} \in V_{\Delta, \varepsilon}(f_{z_k})$ . By (2.2)  $V_{\Delta, \varepsilon}(f_{z_k}) \subset V_{\Delta, \varepsilon(k)}(f_{z_k})$ . Hence  $g_{z_k}$  does not continue beyond  $\Delta$ . Since  $Z$  is dense in  $\partial\Omega$   $g$  is noncontinuable beyond  $\Omega$ .

**3. Generalization of the theorem of Ryll-Nardzewski and Steinhaus.** Siciak observed that the  $n$ -dimensional version of the theorem of Ryll-Nardzewski and Steinhaus may be proved by the same method as that of one variable.

**3.1.** Let  $X$  be a Fréchet space. Given a domain  $\Omega \subset \mathbb{C}^n$ , let  $A$  be a countable subset of  $\Omega$ , dense in  $\Omega$ . Assume that a function  $F: X \times \Omega \rightarrow \mathbb{C}$  satisfies the following conditions

- 1) For any  $x \in X$  the function  $F_x: \Omega \ni \zeta \rightarrow F(x, \zeta) \in \mathbb{C}$  is analytic in  $\Omega$ .
- 2) For any  $\zeta \in \Omega$  the function  $F_\zeta: X \ni x \rightarrow F(x, \zeta) \in \mathbb{C}$  is linear and continuous in  $X$ .

We say that a point  $(a, p) \in A \times \mathbb{Q}$  is regular with respect to  $F$  if  $p > \text{dist}(a, \partial\Omega)$  and  $\varrho(T_a F_x) \geq p$  for all  $x \in X$ .

Let  $R$  denote the set of all pairs  $(a, p) \in A \times \mathbb{Q}$  regular with respect to  $F$ . Set

$$T := \{(a, p) \in (A \times \mathbb{Q}) \setminus R: p > \text{dist}(a, \partial\Omega)\}.$$

We define

$$M_\nu(a, p) := \{x \in X: \varrho(T_a F_x) \geq p, \|T_a F_x\|_{B(a, p)} \leq \nu\}, \quad \nu \in \mathbb{N}, (a, p) \in A \times \mathbb{Q},$$

where

$$\|T_a F_x\|_{B(a, p)} = \sup\{|T_a F_x(\zeta)|, \zeta \in B(a, p)\};$$

$$\mathcal{P} := \bigcup_{\nu=1}^{\infty} \bigcup_{(a, p) \in T} M_\nu(a, p), \quad \mathcal{Q} := X \setminus \mathcal{P};$$

$$\mathcal{G} := \bigcup_{(a, p) \in R} B(a, p) \cap \partial\Omega, \quad \mathcal{H} := (\partial\Omega) \setminus \mathcal{G}.$$

With this denotations we have

**3.2. THEOREM** (for  $n = 1$  Theorem 4.3.1 in [2]; comp. Theorem 4 in [10], Theorem 10.1 in [12] and § 4.5 in [7]).

- (i) The set  $\mathcal{P}$  is of the first category in  $X$ .
- (ii) For every  $\zeta \in \mathcal{G}$  and  $x \in X$  there exists a point  $a \in A$  such that  $\varrho(T_a F_x) > \|a - \zeta\|$ .
- (iii) For every  $\zeta \in \mathcal{H}$ ,  $x \in \mathcal{Q}$ ,  $a \in A$ ,  $\varrho(T_a F_x) \leq \|a - \zeta\|$ .

**Proof.** (i) By virtue of the Vitali Theorem the sets  $M_\nu(a, p)$  are closed in  $X$ . So it suffices to prove that  $\text{int} M_\nu(a, p)$  is empty if  $(a, p) \in T$ ,  $\nu \in \mathbb{N}$ . Suppose it is not true and choose  $\nu \in \mathbb{N}$  and  $(a, p) \in T$  such that  $\text{int} M_\nu(a, p) \neq \emptyset$ . Let  $x_0 \in \text{int} M_\nu(a, p)$ . By definition of  $T$  we can find a point  $x \in X$  such that  $\varrho(T_a F_x) < p$ . For a suitable  $r > 0$  we have  $x_0 + rx \in \text{int} M_\nu(a, p)$ . As a consequence of 2) we receive  $F(x, \zeta) = \frac{1}{r} [F(x_0 + rx, \zeta) - F(x_0, \zeta)]$ ,

$\zeta \in \Omega$ . Hence  $\varrho(T_a F_x) \geq p$ . Contradiction.

(ii) This part of the theorem is an immediate consequence of the definition of the set  $\mathcal{G}$ .

(iii) Suppose that there exist  $\zeta \in \mathcal{H}$ ,  $x \in \mathcal{Q}$  and  $a \in A$  such that  $\varrho(T_a F_x) > \|a - \zeta\|$ . Then we can choose such numbers  $p \in Q$  and  $v \in N$  that  $x \in M_v(a, p)$ ,  $\zeta \in B(a, p)$ . Since  $x \in \mathcal{Q}$  and  $\zeta \in \partial\Omega$  the pair  $(a, p)$  belongs to  $R$ . Hence  $\zeta \in \mathcal{G}$ . But this is in contradiction with hypothesis ( $\zeta \in \mathcal{H}$ ). The proof is concluded.

Let  $\Omega$  be a domain in  $C^n$ . Let us consider the Fréchet space  $\mathcal{O}_\Omega$  with the topology of uniform convergence on compact subsets of  $\Omega$ . From Theorem 3.2 we derive

**3.3. COROLLARY.** *Let  $\Omega$  be a domain in  $C^n$  and let  $A$  be a countable dense subset of  $\Omega$ . Assume that for every  $\zeta \in \partial\Omega$  there exists a function  $g \in \mathcal{O}_\Omega$  such that  $\varrho(T_a g) \leq \|\zeta - a\|$ ,  $a \in A$ . Then  $\Omega$  is the domain of holomorphy and the set  $\{g \in \mathcal{O}_\Omega: g \text{ is noncontinuable beyond } \Omega\}$  is of the second category in  $\mathcal{O}_\Omega$ .*

*Proof.* Put  $X = \mathcal{O}_\Omega$ ,  $F: X \times \Omega \ni (g, \zeta) \rightarrow g(\zeta) \in C$ . Then  $F$  satisfies the hypothesis of Theorem 3.2. In the case under consideration the set  $\mathcal{G}$  is empty. Hence  $\mathcal{H} = \partial\Omega$  and by (iii) of Theorem 3.2 all functions belonging to  $\mathcal{Q}$  are noncontinuable beyond  $\Omega$ .

**3.4.** Given a bounded balanced domain of holomorphy  $\Omega \subset C^n$ , let us consider the two following Banach spaces:

$$X_1 := \left\{ f = \sum_{v=0}^{\infty} f_v: f_v \in P^v(C^n, C), v = 0, 1, \dots; \sum_{v=0}^{\infty} \|f_v\|_\Omega < \infty \right\},$$

$$X_2 := \left\{ f = \sum_{v=0}^{\infty} f_v: f_v \in P^v(C^n, C), v = 0, 1, \dots; \sup_v \|f_v\|_\Omega < \infty \right\}$$

with norms

$$\|f\|_1 = \sum_{v=0}^{\infty} \|f_v\|_\Omega, \quad f \in X_1$$

and

$$\|f\|_2 = \sup_v \|f_v\|_\Omega, \quad f \in X_2.$$

**3.5. THEOREM.** *The following conditions are equivalent*

- 1) *There exists a series  $f \in X_1$  that converges in  $\Omega$  to a function noncontinuable beyond  $\Omega$ .*
- 2) *There exists a series  $f \in X_2$  that converges in  $\Omega$  to a function noncontinuable beyond  $\Omega$ .*
- 3) *There exists a family  $P$  of homogeneous polynomials such that*

$$\Omega = \text{int} \{z \in C^n: |p(z)| < 1, p \in P\}.$$

*Proof.* 1)  $\Rightarrow$  2). This implication is obvious. 2)  $\Rightarrow$  3). Suppose that  $f \in X_2$  is noncontinuable beyond  $\Omega$ . Set

$$P_f := \left\{ \frac{1}{\|f_v\|_\Omega} f_v: v \in N, f_v \neq 0 \right\},$$

$$\Omega' := \text{int} \{z \in C^n: |p(z)| < 1, p \in P_f\}.$$

