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A Difference Analogue of a Non-linear Parabolic Differential Inequalities with Non-linear Boundary Conditions

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In this paper we consider a difference analogue of differential inequalities of the type

$$F(t, x, u, u_x, u_{xx}, u_t) \geqslant F(t, x, v, v_x, v_{xx}, v_t) \tag{0.1}$$

where $(t, x) \in D = [0, T) \times (0, X)^n$; $u_x = (u_{x_1}, \ldots, u_{x_n})$, $u_{xx} = (u_{x_i x_j})$ is an symmetric matrix, with boundary inequalities of the type

$$u \leqslant v \quad \text{for } t = 0 \tag{0.2}$$

$$\Phi_i(t, x, u, u_{x_i}) \leqslant \Phi_i(t, x, v, v_{x_i}) \quad \text{for } x_i = 0 \ (i = 1, \ldots, n) \tag{0.3}$$

$$\Psi_i(t, x, u, u_{x_i}) \geqslant \Psi_i(t, x, v, v_{x_i}) \quad \text{for } x_i = X \ (i = 1, \ldots, n). \tag{0.4}$$

In the second part of this paper the result is transferred to a weakly coupled system of inequalities. The result obtained is an important generalization of Malec result [1]. An analogous result for differential inequalities with boundary conditions of the Dirichlet type has been obtained by Yoshida [2].

1. We denote by E the set $D \times R^{2+n+n^2}$ and assume that F is the function defined for $(t, x, z, q, w, p) \in E$, where $(t, x) \in D$, $z \in R$, $q = (q_1, \ldots, q_n) \in R^n$, $w = (w_{ij}) \in R^{n^2}$, $p \in R$, satisfying the conditions

$$\begin{aligned} F(t, x, z, q, w, p) - F(t, x, \bar{z}, \bar{q}, \bar{w}, \bar{p}) \\ \leqslant \alpha(z-\bar{z}) + \sum_{i=1}^{n} \beta_i(q_i - \bar{q}_i) + \sum_{i,j=1}^{n} \gamma_{ij}(w_{ij} - \bar{w}_{ij}) + \varkappa(p - \bar{p}) \end{aligned} \tag{1.1}$$

for any $(t, x) \in D$; $z, \bar{z} \in R$; $q, \bar{q} \in R^n$; $p, \bar{p} \in R$; $w, \bar{w} \in R^{n^2}$; $p \geqslant \bar{p}$, where $\alpha, \beta_i, \gamma_{ij}, \varkappa$ are functions defined in $D \times R^{2(2+n+n^2)}$ and such that

$$\gamma_{ij} = \gamma_{ji} \quad (i, j = 1, \ldots, n), \qquad \varkappa < 0 \tag{1.2}$$

and γ_{ij} (for $i, j = 1, \ldots, n$; $i \neq j$) is always non-negative or always non-positive.

We introduce the definitions of sets

$$E_i^+ = \{(t, x, z, q_i): t \in (0, T], \ x \in [0, X]^n, \ z \in R, \ q_i \in R, \ x_i = 0\},$$
$$E_i^- = \{(t, x, z, q_i); t \in (0, T], \ x \in [0, X]^n, \ z \in R, \ q_i \in R, \ x_i = X\}$$

$(i = 1, \ldots, n)$

and assume that Φ_i, Ψ_i $(i = 1, \dots, n)$ are functions defined in E_i^+ and E_i^+, respectively, satisfying the conditions

$$\Phi_i(t, x, z, q_i) - \Phi_i(t, x, z, q_i) \geq \delta_i(z-\bar{z}) + \varrho_i(q_i - \bar{q}_i) \tag{1.3}$$

$$\Psi_i(t, x, z, q_i) - \Psi_i(t, x, z, q_i) \leq \varepsilon_i(z-\bar{z}) + \sigma_i(q_i - \bar{q}_i) \tag{1.4}$$

for any $(t, x) \in (0, T] \times [0, X]^n$; $z, \bar{z}, q_i, \bar{q}_i \in R$; $z > \bar{z}$, $(i = 1, \dots, n)$, where δ_i, ϱ_i and ε_i, σ_i are functions defined in $E_i^+ \times R$ and $E_i^- \times R$, respectively, and such that

$$\varrho_i \leq 0, \sigma_i \geq 0 \quad (i = 1, \dots, n). \tag{1.5}$$

2. For fixed natural numbers N_0, N we define $k = T/N_0$, $h = X/N$. Moreover, for a multiindex $m = (m_1, \dots, m_n)(0 \leq m_i \leq N,\ i = 1, \dots, n)$ we define $x_m = (x_1^{m_1}, \dots, x_n^{m_n})$, where $x_i^{m_i} = m_i h$ $(i = 1, \dots, n)$ and $t_j = jk$ $(j = 0, \dots, N_0)$.

We introduce the definitions of the sets

$$Z = \{m = (m_1, \dots, m_n):\ 0 \leq m_i \leq N\ (i = 1, \dots, n)\}$$

$$Z_0 = \{m \in Z:\ 1 \leq m_i \leq N-1\ (i = 1, \dots, n)\}$$

$$\left.\begin{aligned} Z_i^+ &= \{m \in Z:\ m_i = 0 \text{ and } m_j \neq 0, N \text{ for } 1 \leq j < i\} \\ Z_i^- &= \{m \in Z:\ m_i = N \text{ and } m_j \neq 0, N \text{ for } 1 \leq j < i\} \end{aligned}\right\} \quad (i = 1, \dots, n).$$

It is simple to check that $Z_0 \cup \bigcup_{i=1}^{n} Z_i^+ \cup \bigcup_{i=1}^{n} Z_i^- = Z$ and the sets Z_0, Z_i^+, Z_i^- are pairwise separate.

By S we denote the set

$$S = \{(t_j, x_m):\ j = 0, \dots, N_0;\ m \in Z\}.$$

If φ is a function defined in the set S then by φ_m^j we denote its value at the point (t_j, x_m). Then we introduce the denotations

$$i(m) = (m_1, \dots, m_{i-1}, m_i+1, m_{i+1}, \dots, m_n),$$

$$-i(m) = (m_1, \dots, m_{i-1}, m_i-1, m_{i+1}, \dots, m_n)$$

for $m_0 \in Z_0$ $(i = 1, \dots, n)$

and we define the following difference quotients

$$(\varphi_m^j)_t = \frac{1}{k}[\varphi_m^{j+1} - \varphi_m^j] \quad \text{for } j = 0, \dots, N_0 - 1;\ m \in Z$$

$$\left.\begin{aligned} (\varphi_m^j)_{\bar{x}_i} &= \frac{1}{h}[\varphi_{i(m)}^j - \varphi_m^j] \quad \text{for } j = 0, \dots, N_0;\ m \in Z_i^+ \\ (\varphi_m^j)_{\underline{x}_i} &= \frac{1}{h}[\varphi_m^j - \varphi_{-i(m)}^j] \quad \text{for } j = 0, \dots, N_0;\ m \in Z_i^- \\ (\varphi_m^j)_{x_i} &= \frac{1}{2h}[\varphi_{i(m)}^j - \varphi_{-i(m)}^j] \quad \text{for } j = 0, \dots, N_0;\ m \in Z_0 \end{aligned}\right\} \quad (i = 1, \dots, n)$$

$$(\varphi_m^j)_{x_i x_l} = \frac{s_{il}}{2h^2}[-\varphi_{i(m)}^j - \varphi_{l(m)}^j - \varphi_{-i(m)}^j - \varphi_{-l(m)}^j + 2\varphi_m^j + \varphi_{i(s_{il}l(m))}^j + \varphi_{-i(s_{il}l(m))}^j] \quad (i, l = 1, \ldots, n)$$

for $j = 0, \ldots, N_0$; $m \in Z_0$,

where

$$s_{il} = \begin{cases} -1 & \text{for } i = l \text{ or } \gamma_{il} \leqslant 0 \quad (i \neq l) \\ 1 & \text{for } \gamma_{il} > 0 \quad (i \neq l) \end{cases}$$

and denote

$$(\varphi_m^j)_x = ((\varphi_m^j)_{x_1}, \ldots, (\varphi_m^j)_{x_n}), \; (\varphi_m^j)_{xx} = ((\varphi_m^j)_{x_i x_l}).$$

3. THEOREM 1. *If the functions F, Φ_i, Ψ_i $(i = 1, \ldots, n)$ satisfy the assumption of Item 1 and the numbers N_0, N are such that mesh sizes k, h satisfy the inequalities*

(3.1) $$1 - \frac{\alpha k}{\varkappa} + \frac{2k}{\varkappa h^2} \sum_{i=1}^{n} \gamma_{ii} \geqslant 0$$

(3.2) $$\frac{1}{h}\Big(\gamma_{ii} - \sum_{j \neq i} |\gamma_{ij}|\Big) \geqslant \frac{|\beta_i|}{2} \quad (i = 1, \ldots, n)$$

(3.3) $$\delta_i h - \varrho_i > 0, \; \varepsilon_i h + \sigma_i < 0 \quad (i = 1, \ldots, n)$$

and if for any functions u and v

(3.4) $$F(t_j, x_m, u_m^j, (u_m^j)_x, (u_m^j)_{xx}, (u_m^j)_t) \geqslant F(t_j, x_m, v_m^j, (v_m^j)_x, (v_m^j)_{xx}, (v_m^j)_t)$$

for $j = 0, \ldots, N_0 - 1$; $m \in Z_0$

(3.5) $$\Phi_i(t_j, x_m, u_m^j, (u_m^j)_{\bar{x}_i}) \leqslant \Phi_i(t_j, x_m, v_m^j, (v_m^j)_{\bar{x}_i})$$

for $j = 1, \ldots, N_0$; $m \in Z_i^+$ $(i = 1, \ldots, n)$

(3.6) $$\Psi_i(t_j, x_m, u_m^j, (u_m^j)_{x_i}) \geqslant \Psi_i(t_j, x_m, v_m^j, (v_m^j)_{x_i})$$

for $j = 1, \ldots, N_0$; $m \in Z_i^-$ $(i = 1, \ldots, n)$

(3.7) $$u_m^0 \leqslant v_m^0 \quad \text{for } m \in Z$$

then

$$u_m^j \leqslant v_m^j \quad \text{for } j = 0, \ldots, N_0;\; m \in Z.$$

Proof. We denote by $\mathscr{E}$ the set

$$\mathscr{E} = \{(t_j, x_m):\; u_m^j > v_m^j\}.$$

Let us suppose that $\mathscr{E}$ is nonempty. Then, there exist a minimal $j = \bar{j}$ and $m_0 = (m_1^0, \ldots, m_n^0) \in Z$ such that $u_{m_0}^{\bar{j}} > v_{m_0}^{\bar{j}}$. By (3.7) $\bar{j} \geqslant 1$. Put $\bar{j} = j_0 + 1$ where $j_0 \geqslant 0$. We denote $r_m^j = u_m^j - v_m^j$. From here we have

$$r_m^{j_0+1} > 0 \tag{3.8}$$

and

$$r_m^j \leqslant 0 \quad \text{for } j = 0, \ldots, j_0;\ m \in Z. \tag{3.9}$$

By (3.8), (3.9)

$$(u_{m_0}^{j_0})_t - (v_{m_0}^{j_0})_t = (r_{m_0}^{j_0})_t = \frac{1}{k}(r_{m_0}^{j_0+1} - r_{m_0}^{j_0}) > 0$$

i.e.

$$(u_{m_0}^{j_0})_t > (v_{m_0}^{j_0})_t. \tag{3.10}$$

Suppose $m_0 \in Z_0$. For (j_0, m_0) from (1.1) and (3.4) by (3.10) we get

$$0 \leqslant F\big(t_{j_0}, x_{m_0}, u_{m_0}^{j_0}, (u_{m_0}^{j_0})_x, (u_{m_0}^{j_0})_{xx}, (u_{m_0}^{j_0})_t\big) - F\big(t_{j_0}, x_{m_0}, v_{m_0}^{j_0}, (v_{m_0}^{j_0})_x, (v_{m_0}^{j_0})_{xx}, (v_{m_0}^{j_0})_t\big)$$

$$\leqslant \alpha[u_{m_0}^{j_0} - v_{m_0}^{j_0}] + \sum_{i=1}^{n} \beta_i[(u_{m_0}^{j_0})_{x_i} - (v_{m_0}^{j_0})_{x_i}] + \sum_{i,j=1}^{n} \gamma_{ij}[(u_{m_0}^{j_0})_{x_i x_j} - (v_{m_0}^{j_0})_{x_i x_j}]$$

$$+ \varkappa[(u_{m_0}^{j_0})_t - (v_{m_0}^{j_0})_t] = \alpha r_{m_0}^{j_0} + \sum_{i=1}^{n} \beta_i (r_{m_0}^{j_0})_{x_i} + \sum_{i,j=1}^{n} \gamma_{ij}(r_{m_0}^{j_0})_{x_i x_j} + \varkappa(r_{m_0}^{j_0})_t.$$

We rewrite the above inequality in the following form

$$\frac{-\varkappa}{k}[r_{m_0}^{j_0+1} - r_{m_0}^{j_0}] \leqslant \alpha r_{m_0}^{j_0} + \frac{1}{2h}\sum_{i=1}^{n} \beta_i[r_{i(m_0)}^{j_0} - r_{-i(m_0)}^{j_0}]$$

$$+ \frac{1}{h^2}\sum_{i=1}^{n} \gamma_{ii}[r_{i(m_0)}^{j_0} - 2r_{m_0}^{j_0} + r_{-i(m_0)}^{j_0}] + \frac{1}{2h^2}\sum_{i \neq j} |\gamma_{ij}|[-r_{i(m_0)}^{j_0} - r_{j(m_0)}^{j_0}$$

$$- r_{-i(m_0)}^{j_0} - r_{-j(m_0)}^{j_0} + 2r_{m_0}^{j_0} + r_{i(s_{ij}j(m_0))}^{j_0} + r_{-i(-s_{ij}j(m_0))}^{j_0}].$$

Grouping the terms and using (1.2) we obtain

$$r_m^{j_0+1} \leqslant \left(1 - \frac{\alpha k}{\varkappa} + \frac{2k}{\varkappa h^2}\sum_{i=1}^{n} \gamma_{ii}\right) r_{m_0}^{j_0} - \frac{k}{\varkappa h}\sum_{i=1}^{n}\left[\frac{\beta_i}{2} + \frac{1}{h}\left(\gamma_{ii} - \sum_{j \neq i} |\gamma_{ij}|\right)\right] r_{i(m_0)}^{j_0}$$

$$- \frac{k}{\varkappa h}\sum_{i=1}^{n}\left[-\frac{\beta_i}{2} + \frac{1}{h}\left(\gamma_{ii} - \sum_{j \neq i} |\gamma_{ij}|\right)\right] r_{-i(m_0)}^{j_0}$$

$$- \frac{k}{2\varkappa h^2}\sum_{j \neq i} |\gamma_{ij}|[2r_{m_0}^{j_0} + r_{i(s_{ij}j(m_0))}^{j_0} + r_{-i(-s_{ij}j(m_0))}^{j_0}]$$

whence by (3.1), (3.2), (1.2) and (3.9) we get

$$r_{m_0}^{j_0+1} \leqslant 0$$

which contradicts (3.8). Hence $m_0 \notin Z_0$ and

(3.11) $$r_{m_0}^{j_0+1} \leqslant 0 \quad \text{for } m \in Z_0 .$$

Since $m_0 \notin Z_0$ there exists i_0 such that $m_0 \in Z_{i_0}^+$ or $m_0 \in Z_{i_0}^-$. We denote

$$k_0 = \begin{cases} i_0 & \text{when } m_0 \in Z_{i_0}^+ \\ -i_0 & \text{when } m_0 \in Z_{i_0}^- \end{cases}$$

If $k_0 = i_0$ then by (3.5) and (1.3) for $(\tilde{j}, m_0)$ we get (as $u_{m_0}^{\tilde{j}} > v_{m_0}^{\tilde{j}}$, cf. (3.8))

$$0 \geqslant \Phi_{i_0}(t_{\tilde{j}}, x_{m_0}, u_{m_0}^{\tilde{j}}, (u_{m_0}^{\tilde{j}})_{\tilde{x}_{i_0}}) - \Phi_{i_0}(t_{\tilde{j}}, x_{m_0}, v_{m_0}^{\tilde{j}}, (v_{m_0}^{\tilde{j}})_{\tilde{x}_{i_0}})$$
$$\geqslant \delta_{i_0}(u_{m_0}^{\tilde{j}} - v_{m_0}^{\tilde{j}}) + \varrho_{i_0}[(u_{m_0}^{\tilde{j}})_{\tilde{x}_{i_0}} - (v_{m_0}^{\tilde{j}})_{\tilde{x}_{i_0}}] = \delta_{i_0} r_{m_0}^{\tilde{j}} + \varrho_{i_0}(r_{m_0}^{\tilde{j}})_{\tilde{x}_{i_0}}$$
$$= \delta_{i_0} r_{m_0}^{\tilde{j}} + \frac{\varrho_{i_0}}{h}(r_{i_0(m_0)}^{\tilde{j}} - r_{m_0}^{\tilde{j}}) .$$

Rewriting the above inequality in the equivalent form we have (in view of (3.3))

(3.12) $$r_{m_0}^{\tilde{j}} \leqslant \frac{-\varrho_{i_0}}{\delta_{i_0} h - \varrho_{i_0}} r_{k_0(m_0)}^{\tilde{j}} .$$

If $k_0 = -i_0$ by (3.6), (1.5), (3.3) we obtain, in a similar way,

(3.13) $$r_{m_0}^{\tilde{j}} \leqslant \frac{-\sigma_{i_0}}{\varepsilon_{i_0} h + \sigma_{i_0}} r_{k_0(m_0)}^{\tilde{j}} .$$

We denote

$$d_{i_0} = \begin{cases} \dfrac{-\varrho_{i_0}}{\delta_{i_0} h - \varrho_{i_0}} & \text{when } k_0 = i_0 \\[2ex] \dfrac{-\sigma_{i_0}}{\varepsilon_{i_0} h + \sigma_{i_0}} & \text{when } k_0 = -i_0 . \end{cases}$$

By (1.5), (3.3) we have

(3.14) $$d_{i_0} \geqslant 0$$

and by (3.12), (3.13) we have

(3.15) $$r_{m_0}^{\tilde{j}} \leqslant d_{i_0} r_{k_0(m_0)}^{\tilde{j}} .$$

If $r_{k_0(m_0)}^{\tilde{j}} \leqslant 0$ then from (3.14) and (3.15) it follows that $r_{m_0}^{\tilde{j}} \leqslant 0$ which contradicts (3.8). Hence $r_{k_0(m_0)}^{\tilde{j}} > 0$. Since for $m \in Z_0$ (by (3.11)) $r_m^{\tilde{j}} \leqslant 0$, $k_0(m_0) \notin Z_0$. Then there exists i_1 such that $k_0(m_0) \in Z_{i_1}^+$ or $k_0(m_0) \in Z_{i_1}^-$. From the definitions of the sets Z_i^+, Z_i^- it follows that $i_1 > i_0$. Denoting

$$k_1 = \begin{cases} i_1 & \text{when } k_0(m_0) \in Z_{i_1}^+ \\ -i_1 & \text{when } k_0(m_0) \in Z_{i_1}^- \end{cases}$$

and

$$d_{i_1} = \begin{cases} \dfrac{-\varrho_{i_1}}{\delta_{i_1}h-\varrho_{i_1}} & \text{when } k_1 = i_1 \\[2ex] \dfrac{-\sigma_{i_1}}{\varepsilon_{i_1}h+\sigma_{i_1}} & \text{when } k_i = -i_1 \end{cases}$$

we have (as above) $d_{i_1} \geqslant 0$ and

$$\tilde{r}^{j}_{k_0(m_0)} \leqslant d_{i_1} \tilde{r}^{j}_{k_1(k_0(m_0))} .$$

Similarly as above, $\tilde{r}^{j}_{k_1(k_0(m_0))} \leqslant 0$, which contradicts (3.8). Hence $\tilde{r}^{j}_{k_1(k_0(m_0))} > 0$ and then there exists $i_2 > i_1$ such that $k_1(k_0(m_0)) \in Z^{+}_{i_2}$ or $k_1(k_0(m_0)) \in Z^{-}_{i_2}$. We proceed with that process and obtain the strongly increasing sequence of indices $i_0, i_1, i_2, \ldots$ and the corresponding sequences $k_0, k_1, k_2, \ldots$; $d_{i_0}, d_{i_1}, d_{i_2}, \ldots$ such that

$$\tilde{r}^{j}_{m_0} \leqslant \Big(\prod_{l=0}^{n} d_{i_l}\Big) \tilde{r}^{j}_{k_s(\ldots k_1(k_0(m_0))\ldots)} \quad \text{for } s = 0, 1, 2, \ldots$$

and

$$d_{i_s} \geqslant 0, \; \tilde{r}^{j}_{k_s(\ldots k_1(k_0(m_0))\ldots)} > 0 \quad \text{for } s = 0, 1, 2, \ldots$$

As the sequence $i_0, i_1, i_2, \ldots$ is infinite and strongly increasing, there exists s_0 such that $i_{s_0} > n$ and then there exists the set $Z^{+}_{i_{s_0}}$ or $Z^{-}_{i_{s_0}}$ for $i_{s_0} > n$, which is false. Hence the set $\mathscr{E}$ is empty. Therefore

$$u^{j}_{m} \leqslant v^{j}_{m} \quad \text{for } j = 0, \ldots, N_0; \; m \in Z .$$

This ends the proof.

Malec presents in his paper [1] a shorter proof of a similar theorem in the case $m_0 \notin Z$, but it is not correct.

4. Assume F^{ν} $(\nu = 1, \ldots, M)$ are the functions defined for $(t, x, z, q, w, p) \in \hat{E} = D \times R^{M+1+n+n^2}$, where $(t, x) \in D$, $z = (z_1, \ldots, z_M) \in R^M$, $q = (q_1, \ldots, q_n) \in R^n$, $w = (w_{ij}) R^{n^2}$, $p \in R$ satisfying the condition

(4.1)
$$F^{\nu}(t, x, z, q, w, p) - F^{\nu}(t, x, \bar{z}, \bar{q}, \bar{w}, \bar{p}) \leqslant \sum_{i=1}^{M} \alpha^{\nu}_{i}(z_i - \bar{z}_i) + \sum_{i=1}^{n} \beta^{\nu}_{i}(q_i - \bar{q}_i) + \sum_{i,j=1}^{n} \gamma^{\nu}_{ij}(w_{ij} - \bar{w}_{ij}) + \varkappa^{\nu}(p - \bar{p})$$

for any $(t, x) \in D$; $z, \bar{z} \in R^M$; $p, \bar{p} \in R$; $p \geqslant \bar{p}$ $(\nu = 1, \ldots, M)$, where $\alpha^{\nu}_{i}, \beta^{\nu}_{i}, \gamma^{\nu}_{ij}, \varkappa^{\nu}$ are the functions defined in $D \times R^{2(M+1+n+n^2)}$ and such that

(4.2) $$\gamma^{\nu}_{ij} = \gamma^{\nu}_{ji} \; (i, j = 1, \ldots, n; \; \nu = 1, \ldots, M); \; \varkappa^{\nu}_{d} < 0 \; (\nu = 1, \ldots, M)$$

(4.3) $$\alpha^{\nu}_{i} \geqslant 0 \; (i = 1, \ldots, M, \; i \neq \nu; \; \nu = 1, \ldots, M)$$

and γ^{ν}_{ij} $(i, j = 1, \ldots, n; i \neq j; \nu = 1, \ldots, M)$ is always non-negative or always non-positive.

We introduce the sets

$$\hat{E}_i^+ = \{(t, x, z, q_i):\ t \in (0, T],\ x \in [0, X]^n,\ z \in R^M,\ q_i \in R,\ x_i = 0\}$$
$$\hat{E}_i^- = \{(t, x, z, q_i):\ t \in (0, T],\ x \in [0, X]^n,\ z \in R^M,\ q_i \in R,\ x_i = X\}$$
$$(i = 1, \ldots, n)$$

and assume that Φ_i^ν, Ψ_i^ν $(i = 1, \ldots, n;\ \nu = 1, \ldots, M)$ are the functions defined in $\hat{E}_i^+$ and $\hat{E}_i^-$, respectively, satysfying the conditions

$$\Phi_i^\nu(t, x, z, q_i) - \Phi_i^\nu(t, x, \bar{z}, \bar{q}_i) \geqslant \sum_{l=1}^{M} \delta_{il}^\nu (z_l - \bar{z}_l) + \varrho_i^\nu (q_i - \bar{q}_i) \tag{4.4}$$

$$\Psi_i^\nu(t, x, z, q_i) - \Psi_i^\nu(t, x, \bar{z}, \bar{q}_i) \leqslant \sum_{l=1}^{M} \varepsilon_{il}^\nu (z_l - \bar{z}_l) + \sigma_i^\nu (q_i - \bar{q}_i) \tag{4.5}$$

$$(i = 1, \ldots, n;\ \nu = 1, \ldots, M)$$

for any $(t, x) \in (0, T] \times [0, X]^n$; $z, \bar{z} \in R^M$; $q_i, \bar{q}_i \in R$; $z_\nu \geqslant \bar{z}_\nu$, where δ_{il}^ν, ϱ_i^ν and ε_{il}^ν, σ_i^ν are the functions defined in $\hat{E}_i^+ \times R^{M+1}$ and $\hat{E}_i^- \times R^{M+1}$, respectively, and such that

$$\varrho_i^\nu \leqslant 0,\ \sigma_i^\nu \geqslant 0\ (i = 1, \ldots, n;\ \nu = 1, \ldots, M) \tag{4.6}$$

$$\delta_{il}^\nu \leqslant 0,\ \varepsilon_{il}^\nu \geqslant 0\ (i = 1, \ldots, n;\ \nu, l = 1, \ldots, M,\ \nu \neq l). \tag{4.7}$$

The sets Z, Z_0, Z_i^+, Z_i^-, S and the difference quotients for the functions ${}^1\varphi, \ldots, {}^M\varphi$ are defined like in Item 2. The index ν in the definition of $({}^\nu\varphi_m^j)_{x_i x_l}$ is associated with the function γ_{ij}^ν.

5. THEOREM 2. *If the functions F^ν $(\nu = 1, \ldots, M)$, Φ_i^ν, Ψ_i^ν $(i = 1, \ldots, n;\ \nu = 1, \ldots, M)$ satisfy the assumption of Item 4 and the numbers N_0, N are such that the mesh sizes k, h satisfy the inequalities*

$$1 - \frac{\alpha_\nu^\nu k}{\varkappa^\nu} + \frac{2k}{\varkappa^\nu h^2} \sum_{i=1}^{n} \gamma_{ii}^\nu \geqslant 0 \quad (\nu = 1, \ldots, M) \tag{5.1}$$

$$\frac{1}{h}\Big(\gamma_{ii}^\nu - \sum_{j \neq i} |\gamma_{ij}^\nu|\Big) \geqslant \frac{|\beta_i^\nu|}{2} \quad (i = 1, \ldots, n;\ \nu = 1, \ldots, M) \tag{5.2}$$

$$\delta_{i\nu}^\nu h - \varrho_i^\nu > 0,\ \varepsilon_{i\nu}^\nu h + \sigma_i^\nu < 0 \quad (i = 1, \ldots, n;\ \nu = 1, \ldots, M) \tag{5.3}$$

and if for any functions ${}^1u, \ldots, {}^Mu$; ${}^1v, \ldots, {}^Mv$

$$F^\nu(t_j, x_m, u_m^j, ({}^\nu u_m^j)_x, ({}^\nu u_m^j)_{xx}, ({}^\nu u_m^j)_t) \geqslant F^\nu(t_j, x_m, v_m^j, ({}^\nu v_m^j)_x, ({}^\nu v_m^j)_{xx}, ({}^\nu v_m^j)_t) \tag{5.4}$$

for $j = 0, \ldots, N_0 - 1$; $m \in Z_0$ $(\nu = 1, \ldots, M)$

where $u_m^j = ({}^1u_m^j, \ldots, {}^Mu_m^j)$,

$$\Phi_i^\nu(t_j, x_m, \overset{*}{u}{}^j_{i(m)}, ({}^\nu u_m^j)_{\bar{x}_i}) \leqslant \Phi_i^\nu(t_j, x_m, \overset{*}{v}{}^j_{i(m)}, ({}^\nu v_m^j)_{\bar{x}_i}) \tag{5.5}$$

for $j = 1, \ldots, N_0$; $m \in Z_i^+$ $(i = 1, \ldots, n;\ \nu = 1, \ldots, M)$

$$\Psi_i^\nu(t_j, x_m, \overset{*}{u}{}^j_{-i(m)}, ({}^\nu u_m^j)_{x_i}) \geqslant \Psi_i^\nu(t_j, x_m, \overset{*}{v}{}^j_{-i(m)}, ({}^\nu v_m^j)_{\bar{x}_i}) \tag{5.6}$$

for $j = 1, \ldots, N_0$; $m \in Z_i^-$ $(i = 1, \ldots, n;\ \nu = 1, \ldots, M)$

where $\overset{*}{u}{}^j_{i(m)} = ({}^1u^j_{i(m)}, \ldots, {}^{i-1}u^j_{i(m)}, {}^iu^j_m, {}^{i+1}u^j_{i(m)}, \ldots, {}^Mu^j_{i(m)})$, $u^j_{-i(m)} = ({}^1u^j_{-i(m)}, \ldots, {}^{i-1}u^j_{-i(m)}, {}^iu^j_m, {}^{i+1}u^j_{-i(m)}, \ldots, {}^Mu^j_{-i(m)})$
and

$${}^\nu u_m^0 \leqslant {}^\nu v_m^0 \quad \textit{for } m \in Z \ (\nu = 1, \ldots, M) \tag{5.7}$$

then

$${}^\nu u_m^j \leqslant {}^\nu v_m^j \quad \textit{for } j = 0, \ldots, N_0;\ m \in Z \ (\nu = 1, \ldots, M).$$

Proof. We denote by $\hat{\mathscr{E}}$ the set

$$\hat{\mathscr{E}} = \{(t_j, x_m) \colon \exists \nu \in \{1, \ldots, M\} \colon {}^\nu u_m^j > {}^\nu v_m^j\}.$$

Let us suppose that $\hat{\mathscr{E}}$ is nonempty, Then there exists a minimal $j = \tilde{j}$, $m_0 = (m_1^0, \ldots, m_n^0)$ and $\bar{\nu}$ $(1 \leqslant \bar{\nu} \leqslant M)$ such that ${}^{\bar\nu}u^{\tilde j}_{m_0} > {}^{\bar\nu}v^{\tilde j}_{m_0}$. By (5.7) $\tilde{j} > 1$. We put $\tilde{j} = j_0 + 1$ where $j_0 \geqslant 0$. Denoting ${}^\nu r_m^j = {}^\nu u_m^j - {}^\nu v_m^j$ we have

$${}^{\bar\nu}r^{j_0+1}_{m_0} > 0 \tag{5.8}$$

and

$${}^\nu r_m^j \leqslant 0 \quad \text{for } j = 0, \ldots, j_0;\ m \in Z;\ \nu = 1, \ldots, M. \tag{5.9}$$

The further part of the proof is analogous to the proof of Theorem 1 and we will present only these fragments where we deal with differences.

If $m \in Z_0$ then by (4.1), (4.2), (5.4) we get

$$\begin{aligned}
{}^{\bar\nu}r^{j_0+1}_{m_0} \leqslant {} & \left(1 - \frac{\alpha^{\bar\nu}_{\bar\nu} k}{\varkappa^{\bar\nu}} + \frac{2k}{\varkappa^{\bar\nu} h^2} \sum_{i=1}^{n} \gamma^{\bar\nu}_{ii}\right) {}^{\bar\nu}r^{j_0}_{m_0} - \frac{k}{\varkappa^{\bar\nu}} \sum_{\substack{i=1\\ i \neq \bar\nu}}^{n} \alpha_i^{\bar\nu}\, {}^i r^{j_0}_{m_0} - \frac{k}{\varkappa^{\bar\nu} h} \sum_{i=1}^{n} \left[\frac{\beta_i^{\bar\nu}}{2}\right. \\
& \left. + \frac{1}{h}\left(\gamma^{\bar\nu}_{ii} - \sum_{j \neq i} |\gamma^{\bar\nu}_{ij}|\right)\right] {}^{\bar\nu}r^{j_0}_{i(m_0)} - \frac{k}{\varkappa^{\bar\nu} h} \sum_{i=1}^{n} \left[-\frac{\beta_i^{\bar\nu}}{2} + \frac{1}{h}\left(\gamma^{\bar\nu}_{ii} - \sum_{j \neq i} |\gamma^{\bar\nu}_{ij}|\right)\right] {}^{\bar\nu}r^{j_0}_{-i(m_0)}, \\
& - \frac{k}{2\varkappa^{\bar\nu} h^2} \sum_{i \neq j} |\gamma^{\bar\nu}_{ij}| [2\, {}^{\bar\nu}r^{j_0}_{m_0} + {}^{\bar\nu}r^{j_0}_{i(s^{\bar\nu}_{ij} j(m_0))} + {}^{\bar\nu}r^{j_0}_{-i(-s^{\bar\nu}_{ij} j(m_0))}]
\end{aligned}$$

whence from (4.2), (4.3), (5.1), (5.2), (5.9) we obtain

$${}^{\bar\nu}r^{j_0+1}_{m_0} \leqslant 0$$

which contradicts (5.8). Hence $m_0 \notin Z_0$ and

$$(5.10) \qquad {}^{v}r_m^{j_0+1} \leqslant 0 \quad \text{for } m \in Z_0;\ v = 1, \ldots, M.$$

Since $m_0 \notin Z_0$ then there is i_0 such that $m_0 \in Z_{i_0}^+$ or $m_0 \in Z_{i_0}^-$. Suppose $m_0 \in Z_{i_0}^+$ (if $m_0 \in Z_{i_0}^-$ the proof is analogous). From (4.4), (5.3), (5.5) like in the proof of Theorem 1 we obtain

$$ {}^{\bar{v}}r_{m_0}^{\tilde{j}} \leqslant \frac{-\varrho_{i_0}^{\bar{v}}}{\delta_{i_0\bar{v}}^{\bar{v}}h-\varrho_{i_0}^{\bar{v}}}\, {}^{\bar{v}}r_{i_0(m_0)}^{\tilde{j}} - \frac{h}{\delta_{i_0\bar{v}}^{\bar{v}}h-\varrho_{i_0}^{\bar{v}}} \sum_{\substack{l=1\\ l\neq\bar{v}}}^{M} \delta_{i_0 l}^{\bar{v}}\, {}^{l}r_{i_0(m_0)}^{\tilde{j}}.$$

If ${}^{v}r_{i_0(m_0)}^{\tilde{j}} \leqslant 0$ for $v = 1, \ldots, M$ then by (4.6), (4.7), (5.3) from the above inequality we get

$$ {}^{\bar{v}}r_{m_0}^{\tilde{j}} \leqslant 0$$

which contradicts (5.8). Then there exists v_1 $(1 \leqslant v_1 \leqslant M)$ such that ${}^{v_1}r_{i_0(m_0)}^{\tilde{j}} > 0$. Because ${}^{v}r_m^{\tilde{j}} \leqslant 0$ for $m \in Z_0$ (by (5.10)) then $i_0(m_0) \notin Z_0$. Proceeding like in the proof of Theorem 1 we get the contradiction hence the set $\hat{\mathscr{E}}$ is empty.

This ends the proof.

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