UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXVI
1987

A Necessary Condition for the Existence of the Extension Operator from the Fréchet Algebra of Whitney Fields on a Locally Closed Subset of $\mathbf{R}^n$

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Whitney's extension theorem [1] provides a continuous linear extension operator from the space of C^m Whitney fields ($m < \infty$) on a locally closed subset E of $\boldsymbol{R}^n$ to the space of C^m functions on an open neighborhood of E. Let $I^m(E)$ denote the space of C^m Whitney fields on E. We can consider it as a commutative ring with identity and ask if there exists an extension operator which is a homomorphism of rings $I^m(E) \to I^m(G)$, where G is an open neighborhood of E. In general even a linear continuous extension operator need not exist for the space of C^∞ Whitney fields on E but in the cases where it exists — for instance when E is a half-space, or a domain with boundary which is locally the graph of a Hölder function, or a semianalytic set with dense interior — we can also study the problem given above.

In this paper we give a necessary condition for the existence of the extension operator which is a homomorphism of rings $I^\alpha(E) \to I^\alpha(G)$, $\alpha \in N \cup \{\infty\}$, for a locally closed subset E of $\boldsymbol{R}^n$, and its open neighborhood G. Then we consider some examples of subsets of $\boldsymbol{R}^n$, which do not satisfy that condition. The proof of our main theorem is based upon the properties of topological rings $C^\alpha(\mathscr{U})$ (where $\mathscr{U}$ is an open subset of $\boldsymbol{R}^n$, $\alpha \in N \cup \{\infty\}$), which are studied in section 1.

1. In this section we deal with topological rings $C^\alpha(\mathscr{U})$, where $\alpha \in N \cup \{\infty\}$, and $\mathscr{U}$ is an open subset of $\boldsymbol{R}^n$. We describe maximal ideals in such rings, and we prove the theorem which characterizes continuous homomorphisms $C^\alpha(\mathscr{U} \to C^\beta(V)$.

Let $\mathscr{U}$ be an open subset of $\boldsymbol{R}^n$. By $C^\alpha(\mathscr{U})$ we denote the set of C^α real functions on $\mathscr{U}$. If considered with addition and multiplication of functions, $C^\alpha(\mathscr{U})$ is a commutative ring with identity.

PROPOSITION 1.1. *If I is a maximal ideal in $C^\alpha(\mathscr{U})$, then either*

(i) *there exists $p \in \mathscr{U}$ such that $I = \{f \in C^\alpha(\mathscr{U}) : f(p) = 0\}$ or*

(ii) *there exists $\mathscr{A}$, a family of closed no compact subsets of $\mathscr{U}$, such that $\mathscr{A}$ is closed under finite operations of union and intersection, and $I = \{f\colon \exists A \in \mathscr{A}: f_A \equiv 0\}$.*

Proof. Let $\mathscr{A}$ be a family of closed subsets of $\mathscr{U}$ closed under finite union and finite intersection. It is easy to check that $I_{\mathscr{A}} \overset{\mathrm{df}}{=} \{f \in C^\alpha(\mathscr{U})\colon \exists A \in \mathscr{A}: f|_A \equiv 0\}$ is an ideal in the ring $C^\alpha(\mathscr{U})$.

If I is an ideal in $C^\alpha(\mathscr{U})$, then $\{f = 0\}_{f\in I}$ is a family of closed subsets of $\mathscr{U}$ closed under finite operations of union and intersection as we have: $\{f = 0\} \cap \{g = 0\} = \{f^2+g^2 = 0\}$, $\{f = 0\} \cup \{g = 0\} = \{f\cdot g = 0\}$. We denote $\{f = 0\}_{f\in I}$ by $\mathscr{N}(I)$.

We shall show that if I is a maximal ideal in $C^\alpha(\mathscr{U})$, then $I = I_{\mathscr{N}(I)}$.

The condition $I \subset I_{\mathscr{N}(I)}$ is an immediate consequence of the definition of $I_{\mathscr{N}(I)}$. Let g be an element of $I_{\mathscr{N}(I)}$. There exists $f \in I$ such that $g|_{\{f=0\}} \equiv 0$. Suppose that $g \notin I$. Maximality of I implies that there exist $k \in I$, $h \in C^\alpha(U)$: $1 \equiv k+hg$, whence $k|_{\{f=0\}} \neq 0$. Since $f^2+k^2 \in I$ we obtain $\emptyset \neq \{f^2+k^2 = 0\} = \{f = 0\} \cap \{k = 0\}$, which leads to the contradiction.

Collecting the results established in previous remarks we obtain: I is a maximal ideal in $C^\alpha(\mathscr{U})$, then there exists a family $\mathscr{A}$ with properties given above, such that $I = I_{\mathscr{A}}$.

Let us suppose that there exists $A_0 \in \mathscr{A}$ such that A_0 is a compact subset of $\mathscr{U}$. Then $\bigcap \mathscr{A} = \bigcup (\mathscr{A} \cap A_0)$ is the intersection of closed subsets of the compact space A_0, which has the finite intersection property. Hence the existence of $A_0 \in \mathscr{A}$ such that A_0 is compact implies that $\bigcap \mathscr{A} \neq \emptyset$. Let $N \overset{\mathrm{df}}{=} \bigcap \mathscr{A}$. For an arbitrary subset E of $\mathscr{U}$ we denote the ideal of the form $\{f\colon f|_E \equiv 0\}$ by I_E. Thus $I = I_{\mathscr{A}} \subset I_N$, whence $I = I_N$ because I is the maximal ideal in $C^\alpha(\mathscr{U})$. Suppose that there exist two distinct points x_1, x_2 belonging to N. Then $I_N \subsetneqq I_{\{x_1\}}$ which contradicts the maximality of $I = I_N$. Hence there exists $x_0 \in \mathscr{U}\colon I = I_{\{x_0\}}$.

If I is the maximal ideal, then $I = I_{\mathscr{A}}$ for a suitable family $\mathscr{A}$. Thus if (i) does not hold, then (ii) holds.

We shall show the existence of maximal ideals such that for every $A \in \mathscr{A}$, A is no compact. To that purpose we shall show an ideal which is not of the form $\{f \in C^\alpha(\mathscr{U})\colon f(p) = 0\}$.

Let $\{x_n\}$ be a sequence of points of $\mathscr{U}$: $\lim x_n = x_0 \in \partial\mathscr{U}$ (if $\partial\mathscr{U} = \emptyset$ we choose a sequence without an accumulate point).

$$A_n \overset{\mathrm{df}}{=} \{x_k\colon k \geqslant n\}, \quad \mathscr{A} \overset{\mathrm{df}}{=} \{A_n\}_{n\in \boldsymbol{N}}, \quad I' \overset{\mathrm{df}}{=} I_{\mathscr{A}}$$

There exists a maximal ideal I, such that $I' \subset I$. We fix $p \in \mathscr{U}$. There exists $n_0 \in \boldsymbol{N}\colon p \notin A_{n_0}$. By the definition of $I_{\mathscr{A}}$ we can find $f \in I$: $\{f = 0\} = A_{n_0}$. This implies that $f(p) \neq 0$, whence $I \neq I_{\{p\}}$.

We shall denote ideals of the form as in proposition 1.1 (i) by I_p, $p \in \mathscr{U}$, ideals of the form as in proposition 1.1 (ii) by I^b.

Let $\mathscr{U}$, V be open subsets of $\boldsymbol{R}^n$, $\boldsymbol{R}^m$ respectively, $\alpha, \beta \in \boldsymbol{N} \cup \{\infty\}$.

Remark 1.2. *If $\lambda\colon C^\alpha(\mathscr{U}) \to C^\beta(V)$ is a homomorphism of rings such that* $\operatorname{Im}\lambda \supset \{f\colon \forall x \in V\, f(x) = c\} \overset{\mathrm{df}}{=} \mathscr{C}$, *then for every $p \in V$, $\lambda^{-1}(I_p)$ is a maximal ideal in $C^\alpha(\mathscr{U})$.*

Proof. For every ideal I contained in $C^{\beta}(V)$, $\lambda^{-1}(I)$ is an ideal in $C^{\alpha}(\mathscr{U})$. Since $\lambda^{-1}(I) = \lambda^{-1}(I \cap \operatorname{Im}\lambda)$, it suffices to show that $\operatorname{Im}\lambda \cap I_p$ is the maximal ideal in $\operatorname{Im}\lambda$.

$$\delta_p\colon \operatorname{Im}\lambda \ni f \to f(p) \in \boldsymbol{R}$$

is the homomorphism of rings mapping $\operatorname{Im}\lambda$ onto $\boldsymbol{R}$, because $\operatorname{Im}\lambda \supset \mathscr{C}$.

$$\operatorname{Ker}\delta_p = I_p \cap \operatorname{Im}\lambda$$

whence

$$\operatorname{Im}\lambda / I_p \cap \operatorname{Im}\lambda \cong \boldsymbol{R}$$

which completes the proof of the remark.

Remark 1.3. *If $\mathscr{A}$ is a family of closed subsets of $\mathscr{U}$ such that $\bigcap \mathscr{A} = \emptyset$, then there exists $\{A_i\}_{i\in \boldsymbol{N}}$, $A_i \in \mathscr{A}$: $\bigcap_{i=1}^{\infty} A_i = \emptyset$.*

Proof. $\{\mathscr{U}\setminus A\}_{A\in\mathscr{A}}$ is the open covering of $\mathscr{U}$. We can choose $\{A_i\}_{i\in \boldsymbol{N}}$, the countable subfamily of $\mathscr{A}$, such that $\{\mathscr{U}\setminus A_i\}_{i\in \boldsymbol{N}}$, is an open covering of $\mathscr{U}$ as well, whence we obtain $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

We consider $C^{\alpha}(\mathscr{U})$ as a topological ring with respect to the the topology of uniform convergence of functions and their partial derivatives on compact sets (for $\alpha = 0$ it means uniform convergence of functions on compact sets). For all α and $\mathscr{U}$, $C^{\alpha}(\mathscr{U})$ is then a complete metric space.

PROPOSITION 1.4.

(i) *For every $p \in \mathscr{U}$, I_p is a closed subset of $C^{\alpha}(\mathscr{U})$.*

(ii) *For every I^b, I^b is not a closed subset of $C^{\alpha}(\mathscr{U})$, if $C^{\alpha}(\mathscr{U})$ and $\bar{I}^b$ are not equal.*

Proof. (i) Since $C^{\alpha}(\mathscr{U})$ is the metric space it suffices to check that if $f_n \in I_p$, $\lim f_n = f$ then $f \in I_p$, which is clear.

(ii) According to the proposition 1.1 $I^b = I_{\mathscr{A}}$, where $\mathscr{A}$ is a family of closed subsets of $\mathscr{U}$, which has the properties mentioned in the proposition 1.1. In particular $\bigcap \mathscr{A} = \emptyset$. We choose $A_i \in \mathscr{A}$, $i \in \boldsymbol{N}$: $\bigcap_{i=1}^{\infty} A_i = \emptyset$ (remark 1.3).

$$B_n \stackrel{\mathrm{df}}{=} \bigcap_{i=1}^{n} A_i, \quad B_n \in \mathscr{A}, \quad B_n \searrow \emptyset$$

Let

$$C_k \stackrel{\mathrm{df}}{=} \left\{x \in \mathscr{U}\colon d(x, B_k) \geqslant \frac{\varepsilon}{2^k}\right\}$$

for an arbitrary fixed real positive number ε. Since $B_m \subset B_n$ for $m \geqslant n$, and $B_n \searrow \emptyset$, there exists $k_0 \in \boldsymbol{N}$ such that for $k \geqslant k_0$, $C_k \neq \emptyset$ and $C_k \cap B_k = \emptyset$. C_k are closed subsets of $\mathscr{U}$, which let us take $f_k \in C^{\alpha}(\mathscr{U})$:

$$f_k|_{B_k} \equiv 0, \quad f_k|_{C_k} \equiv 1$$

$f_k \in I^b = I_{\mathscr{A}}$ because $B_k \in \mathscr{A}$.

We shall show that $\lim f_n = 1$ (1 denotes the identity of the ring $C^\alpha(\mathscr{U})$). This will complete the proof.

For $x_0 \in \mathscr{U}$ fixed, there exists $n_0 \in N$ such that $x_0 \notin B_{n_0}$, because $B_n \searrow \emptyset$. According to the definition of the sets C_k we can find $\beta(n_0) \in N$ and $\delta > 0$:

$$\bar{K}(x_0, \delta) \subset C_m$$

for $m \geqslant \beta(n_0)$. Hence

$$f_m|_{\bar{K}(x_0,\delta)} \equiv 1$$

for $m \geqslant \beta(n_0)$ which means that the sequence f_n converges to 1. As a conclusion of the results established above we have:

PROPOSITION 1.5. *If λ: $C^\alpha(\mathscr{U}) \to C^\beta(V)$ is a continuous (with respect to the suitable topologies) homomorphism of rings, and* $\operatorname{Im}\lambda \supset \mathscr{C}$, *then for every $p \in V$ there exists a unique $q \in \mathscr{U}$ such that $\lambda^{-1}(I_p) = I_q$.*

Proof. For every $p \in V$, $\lambda^{-1}(I_p)$ is the closed maximal ideal in $C^\alpha(\mathscr{U})$. By assumption, $\operatorname{Im}\lambda \supset \mathscr{C}$, which implies that $I_p \cap \operatorname{Im}\lambda \subsetneq \operatorname{Im}\lambda$, whence $\lambda^{-1}(I_p) \subsetneq C^\alpha(\mathscr{U})$. Hence according to the propositions 1.1, 1.4 there exists the unique $q \in \mathscr{U}$ such that $\lambda^{-1}(I_p) = I_q$.

Remark 1.6. We need not assume that $\operatorname{Im}\lambda \supset \mathscr{C}$. This is easy to obtain, for we know that $\lambda(1) = 1$ (under the definition of a homomorphism), and λ is a continuous mapping.

THEOREM 1. *For every continuous homomorphism of rings λ: $C^\alpha(\mathscr{U}) \to C^\beta(V)$, where $\mathscr{U}$, V are the open subsets of $\mathbf{R}^n$, $\mathbf{R}^m$ respectively, $\alpha, \beta \in \mathbf{N} \cup \{\infty\}$, there exists exactly one C^β mapping φ: $V \to \mathscr{U}$ such that $\lambda(f) = f \circ \varphi$ for every $f \in C^\alpha(\mathscr{U})$.*

Proof. Let us suppose that there exists φ: $V \to \mathscr{U}$ such that for every $f \in C^\alpha(\mathscr{U})$,

$$\lambda(f) = f \circ \varphi$$

Then we denote λ by φ^*. Let

$$\pi_i\colon \mathscr{U} \ni x \to x_i \in \mathbf{R}, \quad i = 1, 2, \ldots, n, \quad \pi_i \in C^\alpha(\mathscr{U})$$

If $\varphi = (\varphi_1, \ldots, \varphi_n)$, then

$$\varphi^*(\pi_i) = \varphi_i \in C^\beta(V)$$

This implies that there is at most one φ: $V \to \mathscr{U}$ for which $\lambda = \varphi^*$ holds, and if it exists it is of class C^β. Hence if we find a mapping φ: $V \to \mathscr{U}$ with the property given above, the theorem will be proved.

To that purpose we fix $p \in V$. According to the proposition 1.5 there exists exactly one point $q \in \mathscr{U}$: $\lambda^{-1}(I_p) = I_q$. We define: $q \overset{\text{df}}{=} \varphi(p)$.

Let f, s be fixed elements of $C^\alpha(\mathscr{U})$, V respectively, and $(\lambda(f))(s) = \alpha \in \mathbf{R}$. We denote $\mathscr{U} \ni x \to \alpha \in \mathbf{R}$ by the same letter α.

$$(\lambda(f-a))(s) = 0$$

(the remark after the proof of proposition 1.5 shows that λ is the homomorphism of $\boldsymbol{R}$-algebras $C^{\alpha}(\mathcal{U})$, $C^{\beta}(V)$ i.e. for every $a \in \mathcal{C}$, $\lambda(a) = a$) implies that

$$\lambda(f-a) \in I_s$$

whence $f-a \in I_{\varphi(s)}$. Thus

$$(f \circ \varphi)(s) - (\lambda(f))(s) = 0$$

and we obtain $\lambda(f) = f \circ \varphi$.

PROPOSITION 1.7 *If there exists $\lambda: C^{\alpha}(\mathcal{U}) \to C^{\beta}(V)$, which is the topological isomorphism, then φ, for which $\lambda = \varphi^*$, establishes $C^{\min\{\alpha,\beta\}}$ diffeomorphism between $\mathcal{U}$ and V,*

Proof. It is clear that $\lambda^{-1} = (\varphi^{-1})^*$, and both φ, and φ^{-1} are of class $C^{\min\{\alpha,\beta\}}$. We shall now establish several simple properties of the homomorphisms of the form φ^*.

PROPOSITION 1.8.

(i) *If* $\operatorname{Im}\varphi^* \supset W$: $\forall x, y \in V$, $x \neq y$ $\exists g \in W$: $g(x) \neq g(y)$ *then φ is injective.*

(ii) *If φ^* maps $C^{\alpha}(\mathcal{U})$ onto $C^{\beta}(V)$, then* $\dim V \leqslant \dim \mathcal{U}$.

(iii) *If φ^* is injective, then $\varphi(V)$ is the dense subset of $\mathcal{U}$.*

Proof. Properties (i), (iii) are quite easy to check, so we shall deal only with the case (ii). In general, the formula: $\dim V \leqslant \dim \mathcal{U}$ can be obtained as a result of the Brouwer theorem, which says that the image of an open subset H of $\boldsymbol{R}^n$ by an injective continuous mapping $f: H \to \boldsymbol{R}^n$ is an open subset of $\boldsymbol{R}^n$. If we assume that $\alpha, \beta \geqslant 1$, we may reason as follows: since $\operatorname{Im}\varphi^* \ni \pi_i$, $i = 1, 2, \ldots, m$, and

$$\pi_i = d\pi_{i|V} = dfi \circ d\varphi|_V$$

(for suitable $f_i \in C^{\alpha}(\mathcal{U})$) are linearly independent, it is clear that for every $p \in V$, $\operatorname{rank}_p \varphi = m$. This implies that $\dim V \leqslant \dim \mathcal{U}$.

Remark 1.9. We shall be able to repeat without any changes everything which has been said, if open subsets of $\boldsymbol{R}^n$ are replaced by open subsets of manifolds of the countable type. In particular, proposition 1.7 gives the differential (topological) invariant of a manifold.

2. In this section we prove our main theorem which deals with the extension operators from the Fréchet algebra of Whitney fields on a locally closed subset of $\boldsymbol{R}^n$. Then we give several examples of subsets of $\boldsymbol{R}^n$, which do not satisfy the necessary condition for the existence of such operators.

The space of Whitney fields of class C^m on the locally closed subset E of R^n can be defined as a set of mappings $A: E \to P_m$ (where P_m denotes the space of polynomials of the variables $x_1, \ldots, x_n$, which have the degree equal or less than m), which satisfy the following condition:

$$\forall a \in E \forall p \in N^n, |p| \leqslant m$$

$$r^{m-|p|}_{D^pA}(x, y) = o(|x-y|^{m-|p|}), \quad x, y \to a$$

where:

$$r_A^k(x,y) \overset{\mathrm{df}}{=} a_0(y) - \sum_{|p|\leq k} a_p(x)(y-x)^p$$

$$D^s A\colon E \ni z \to \sum_{p\geq s} a_p(z)\frac{p!}{(p-s)!}x^{p-s}$$

$A(z) \in P_m$ we denote by $\sum_{|s|\leq m} a_s(z)x^s$.

The families of the seminorms:

$$U_K\colon A \to \sup\{|a_p(x)|;\ x\in K,\ |p|\leq m\}$$

$$V_K\colon A \to U_K(A)+\sup\left\{\frac{|r^k_{D^pA}(x,y)|}{|x-y|^k};\ x,y\in K,\ x\neq y,\ |p|\leq m,\ k\leq m-|p|\right\}$$

— where K runs through the family of compact subsets of E — provide two topological structures in $I^m(E)$. Suitable topologies are denoted by τ_0, τ, respectively. $(I^m(E),\tau)$ is always a complete space. For τ_0 and τ to be equal it is necessary and sufficient that $(I^m(E),\tau_0)$ be a complete space. For example it holds if E is an open subset of $\boldsymbol{R}^n$.

If E is an open subset of $\boldsymbol{R}^n$, we have the natural isomorphism of the Fréchet algebras $I^m(E)$, $C^m(E)$ given by the following formula

$$i_E\colon C^m(E)\ni f \to \left([f]\colon E\ni z \to \sum_{|p|\leq m} D^p f(z)\frac{1}{p!}x^p\right)\in I^m(E)$$

For E, F such that $E\subset F$, there exists the homomorphism of the Fréchet algebras $r_E^F\colon I^m(F)\to I^m(E)$ which is defined by: $r_E^F(A) = A|_E$.

The notions and the results given above are similar for the space of C^∞ Whitney fields. In particular we shall use the same symbols i_E, r_E^F for suitable mappings: $C^\infty(E)\to I^\infty(E)$, $I^\infty(F)\to I^\infty(E)$.

THEOREM 2. *Let E be a closed subset of an open subset G of $\boldsymbol{R}^n$. If there exists $\varepsilon\colon (I^\alpha(E),\sigma)\to I^\alpha(G)$, a continuous homomorphism of $\boldsymbol{R}$-algebras (where σ denotes either τ_0 or τ) such that $r_E^G\circ\varepsilon = \mathrm{id}_{I^\alpha(E)}$, then there exists $r\colon G\to G$ of class C^α, which is the retraction G onto E ($\alpha\in \boldsymbol{N}\cup\{\infty\}$).*

Proof. The continuosity of $r_E^G\colon I^\alpha(G)\to I^\alpha(E)$ does not depend of the topology (τ_0 or τ) considered in $I^\alpha(E)$, so by assumption we can define the continuous homomorphism of $\boldsymbol{R}$-algebras (with respect to the ordinary topology) as follows:

$$\begin{cases}\lambda\colon C^\alpha(G)\to C^\alpha(G)\\ \lambda \overset{\mathrm{df}}{=} i_G^{-1}\circ\varepsilon\circ r_E^G\circ i_G\end{cases}$$

According to the theorem 1, there exists $r\colon G\to G$ which is of class C^α and satisfies: $\lambda = r^*$. We shall show that r is the retraction G onto E.

Let $f \in C^\alpha(G)$ be a function which is flat on E, and such that $\{f = 0\} = E$. Under the definition of λ we obtain $\lambda(f) \equiv 0$ whence $f \circ r \equiv 0$. This implies that $r(G) \subset E$. For every $f \in C^\alpha(G)$, $\lambda(f)|_E = f|_E$ which means that

$$f \circ r|_E = f|_E$$

Thus for every $x \in E$, $r(x) = x$, which completes the proof of the theorem 2.

Expressed in other terms, theorem 2 says that for the existence of a continuous extension operator $I^\alpha(E) \to I^\alpha(G)$ which is a homomorphism of rings, it is necessary for E to be the C^α retract of G.

We shall now give two examples of the families of a locally closed sets, which do not satisfy that condition for $\alpha \geqslant 1$.

PROPOSITION 2.1. *If E is a closed subset of an open subset G of $\boldsymbol{R}^n$, such that* $\operatorname{int} E \neq \emptyset$, *then E is not the C^1 retract of G.*

Proof. Let us suppose that there exists $r: G \to G$ the C^1 retraction G onto E. We choose $x_0 \in \partial E$ in such a way that there exists a sequence $\{x_n\}_{n \in \boldsymbol{N}}$, $x_n \in \operatorname{int} E$: $\lim x_n = x_0$. For every n, $d_{x_n} r = \mathrm{id}_{\boldsymbol{R}^n}$, which implies that $d_{x_0} r = \mathrm{id}_{\boldsymbol{R}^n}$. Thus by the local inversion theorem we obtain, that r maps an open neighborhood of x_0 onto another one, which is not possible, because $r(G) \subset E$ and $x_0 \in \partial E$.

In this case we deal with the problem of the extension of an ordinary real function defined on an open subset of $\boldsymbol{R}^n$. Let us suppose that G, H are open subsets of $\boldsymbol{R}^n$ and $\overline{H} \not\subset G$. It can be shown under the assumption of 1-regularity for C^m fields, or that the Whitney condition is satisfied for C^∞ fields, that every function (of a suitable class) for which limits of its partial derivatives exist at the boundary points of H, can be extended onto the set G. The linearity and continuity of such extension can be obtained when the suitable theorem for $I^\alpha(\overline{H})$ holds. According to the proposition 2.1 we can not preserve then the multiplication.

PROPOSITION 2.2. *Let E be a closed subset of an open subset G of $\boldsymbol{R}^n$. If there exists $x_0 \in E$, and $\mathscr{U}$ an open neighborhood of x_0 such that $\mathscr{U} \cap E$ is a C^1 submanifold with boundary* (*of the dimension greater than* 0), *then E is not the C^1 retract of G.*

Proof. Let us suppose that there exists $r: G \to G$ the C^1 retraction G onto E. Under this assumption we can find an open neighborhood W of x_0, the C^1 diffeomorphism $\varphi: W \to \boldsymbol{R}^n$:

$$\varphi(W \cap E) = \{x \in \boldsymbol{R}^n\colon x_k \geqslant 0, x_{k+1} = \ldots = x_n = 0,\ 1 \leqslant k \leqslant n\}$$

Letting

$$R \stackrel{\mathrm{df}}{=} \varphi \circ r \circ \varphi^{-1}$$

we infer that R is the C^1 retraction $\boldsymbol{R}^n$ onto the set

$$\{x \in \boldsymbol{R}^n\colon x_k \geqslant 0,\ x_{k+1} = \ldots = x_n = 0,\ 1 \leqslant k \leqslant n\}$$

This leads to a contradiction (according to the proposition 2.1).

In particular, propositions 2.1, 2.2 deal with the situation where E is a closed subset of $\boldsymbol{R}^n$, and $G = \boldsymbol{R}^n$. If we do not assume the compactness of E, we shall not be able to solve the problem of the retraction of $\boldsymbol{R}^n$ onto E. There exist closed subsets of $\boldsymbol{R}^n$, which are the C^∞ retracts of $\boldsymbol{R}^n$ (for example subspaces), and on the other hand we can find a closed subset which is not even a C^0 retract of $\boldsymbol{R}^n$ for example:

$$\boldsymbol{R}^2 \supset A,\ A \overset{\mathrm{df}}{=} \{x \in \boldsymbol{R}^2\colon x_2 = 0\} \setminus \{x\colon |x_1| < 1\} \cup S^1$$

We do not know whether there exists a compact contained in $\boldsymbol{R}^n$ (not equal to $\{x\}$, $x \in \boldsymbol{R}^n$), which is a C^1 retract of $\boldsymbol{R}^n$. All we can prove is:

PROPOSITION 2.3. If *X contained in $\boldsymbol{R}^n$ is a compact topological manifold of the dimension greater than* 0, *then X is not a C^0 retract of $\boldsymbol{R}^n$.*

Proof. We may assume that X is connected. For the connected, compact topological manifold, the following formula holds:

$$H\dim x(X;\ \boldsymbol{Z}/z) \cong \boldsymbol{Z}/z$$

where $H_k(X;\ G)$ denotes the singular homology group of the topological space X with respect to the group G. It is clear now that the supposition about the existence of the retraction of $\boldsymbol{R}^n$ onto X leads to a contradiction, because the retraction $r\colon A \to B$ provides the epimorphism

$$r_k^*\colon H_k(A;\ G) \to H_k(B;\ G)$$

for every $k \in \boldsymbol{Z}$, and G.

References

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Received March 28, 1981.