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Error estimates for elliptic difference problems

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§ 1. This paper deals with error estimates of a difference method for solving the non-linear system

$$f^l(x, u, u_{lx}, u_{lxx}) = 0 \quad (l = 1, 2, \ldots, p), \tag{1.1}$$

of elliptic type.

In the system (1.1) the l-th equation contains all unknown functions $u = (u_1, u_2, \ldots, u_p)$ and the derivatives of the l-th function u_l only. The functions u_l depend on $x = (x_1, \ldots, x_n)$, u_{lx} denotes a vector and u_{lxx} an $n \times n$ matrix.

In the preceding paper [1] we have given two error estimates for a difference method. The first one is of the form

$$Y_{l1} \leqslant r_l^M \leqslant Y_{l2} \quad (l = 1, \ldots, p), \tag{1.2}$$

cf. [1] (theorem 3), and it seems to be the most precise error estimate. Unfortunately, it is unsatisfactory from the numerical point of view.

The second one has the form

$$|r_l^M| \leqslant \frac{-\varepsilon(h)}{\eta} \quad (l = 1, 2, \ldots, p). \tag{1.3}$$

It is a simple but rather rough estimate since the right-hand member of (1.3) does not depend on the number p, cf. (1.1), of the equations.

This suggests the need for the estimate which is slightly less accurate than (1.2) but is more convenient from the computational point of view.

This error estimate has the form

$$|r_l^M| \leqslant \frac{-\varepsilon(h)}{\eta + (p-1)\delta} \quad (l = 1, \ldots, p)(p \geqslant 2), \tag{1.4}$$

and will be proved in this paper.

The estimate (1.4) is less precise than (1.2) and more accurate than (1.3).

§ 2. In this paper we shall use Assumptions A of the paper [1], (cf. [1], § 4).

§ 3. THEOREM 1. *Let us suppose that the Assumptions A, cf.* [1] (§ 4), *are fulfilled. Under these assumptions*

1° *we have the error estimate*

$$|r_l^M| \leqslant \frac{-\varepsilon(h)}{\eta+(p-1)\cdot\delta} \quad (l = 1, \ldots, p)(p \geqslant 2), \tag{3.1}$$

2° *the three error estimates* (1.2), (1.3) *and* (3.1) *satisfy the relation*

$$\frac{\varepsilon(h)}{\eta} < \frac{\varepsilon(h)}{\eta+(p-1)\delta} \leqslant Y_{l1} \leqslant r_l^{B_l} \leqslant r_l^M \leqslant r_l^{A_l} \leqslant \tag{3.2}$$

$$\leqslant Y_{l2} < \frac{-\varepsilon(h)}{\eta+(p-1)\delta} < \frac{-\varepsilon(h)}{\eta} \quad (l = 1, 2, \ldots, p).$$

Proof. Let us consider the hyperplane π_l:

$$\pi_l\colon \sum_{k=1}^{p} c_{lk}^{A_l}\cdot y_k = -\varepsilon(h) \quad (l = 1, \ldots, p), \tag{3.3}$$

(cf. [1], formula (10.6)).

Let us consider also the hyperplane L_l:

$$L_l\colon \eta\cdot y_l + \sum_{\substack{k=1\\k\neq l}}^{p} c_{lk}^{A_l}\cdot y_k = -\varepsilon(h) \quad (l = 1, \ldots, p), \tag{3.4}$$

(cf. [1], Fig. 2 in the case $p = 2$).

We shall verify that the hyperplane L_l $(l = 1, \ldots, p)$ intersects the straight line $y_1 = y_2 = , \ldots, = y_p$. In fact, the vector $\vec{n}_l$ perpendicular to the hyperplane L_l has the coordinates

$$\vec{n}_l = (n_{l1}, n_{l2}, \ldots, n_{lp}),\ n_{ll} = \eta,\ n_{lk} = c_{lk}^{A_l} \quad (l \neq k). \tag{3.5}$$

Let us take the vector $\vec{e}$ parallel to the straight line $y_1 = y_2 = \ldots = y_p$:

$$\vec{e} = (1, 1, \ldots, 1). \tag{3.6}$$

The scalar product $\vec{n}_l\cdot\vec{e}$ satisfies the inequality

$$\vec{n}_l\cdot\vec{e} = \eta + \sum_{\substack{k=1\\k\neq l}}^{p} c_{lk}^{A_l} < \eta+(p-1)\cdot\delta, \tag{3.7}$$

(cf. [1], formulas (3.5) and (6.3)).

But from the Assumptions A we have

$$\eta+(p-1)\cdot\delta < 0, \tag{3.8}$$

(cf. [1], formula (3.6)), hence

$$\vec{n}_l\cdot\vec{e} < 0 \quad (l = 1, 2, \ldots, p), \tag{3.9}$$

which means, that the hyperplane L_l $(l = 1, \ldots, p)$ intersects the straight line $y_1 = y_2 = \ldots = y_p$.

We shall find now the coordinates of the intersection of the hyperplane L_l $(l = 1, \ldots, p)$ with the straight line $y_1 = y_2 = \ldots = y_p$. To this purpose we substitute $y_1 = y_2 = y_3 = \ldots = y_p$ into the equation (3.4) of the hyperplane and we obtain

$$\text{(3.10)} \qquad \begin{cases} y_1^{(l)} = y_2^{(l)} = \ldots = y_p^{(l)} = \dfrac{-\varepsilon(h)}{\eta + \sum\limits_{\substack{k=1 \\ k \neq l}}^{p} c_{lk}^{A_l}} \\ (l = 1, 2, \ldots, p). \end{cases}$$

Hence, every hyperplane L_l $(l = 1, 2, \ldots, p)$ intersects the straight line $y_1 = y_2 = \ldots = y_p$ at the point with coordinates (3.10).

Let us consider now the following point on the straight line $y_1 = y_2 = \ldots = y_p$:

$$\text{(3.11)} \qquad \tilde{y}_1 = \tilde{y}_2 = \ldots = \tilde{y}_p = d, \; d = \frac{-\varepsilon(h)}{\eta + (p-1)\cdot\delta}.$$

We shall prove that

$$\text{(3.12)} \qquad y_1^{(l)} = y_2^{(l)} = \ldots = y_p^{(l)} < d \quad (l = 1, 2, \ldots, p).$$

In fact, from (3.8) and (3.7) we have

$$\text{(3.13)} \qquad \eta + \sum_{\substack{k=1 \\ k \neq l}}^{p} c_{lk}^{A_l} < \eta + (p-1)\delta < 0.$$

Dividing (3.13) by negative numbers we get

$$\text{(3.14)} \qquad \frac{1}{\eta + (p-1)\delta} < \frac{1}{\eta + \sum\limits_{\substack{k=1 \\ k \neq l}}^{p} c_{lk}^{A_l}} < 0.$$

But $\varepsilon(h) > 0$, hence from (3.14) it follows

$$\text{(3.15)} \qquad \frac{-\varepsilon(h)}{\eta + (p-1)\cdot\delta} > \frac{-\varepsilon(h)}{\eta + \sum\limits_{\substack{k=1 \\ k \neq l}}^{p} c_{lk}^{A_l}} > 0,$$

and this ends the proof of (3.12), because of the formula (3.10).

Let us now pass the hyperplane (3.16) through the point (3.11) parallel to the hyperplane π_l $(l = 1, 2, \ldots, p)$:

$$\text{(3.16)} \qquad \sum_{k=1}^{p} c_{lk}^{A_l} \cdot (y_k - \tilde{y}_k) = 0 \quad (l = 1, 2, \ldots, p).$$

In the paper [1] we have proved that the point $(r_1^{A_1}, \ldots, r_p^{A_p})$ is in the set bounded by the hyperplanes π_l $(l = 1, 2, \ldots, p)$, (cf. [1], formula (10.8) and Fig. 2 in the case $p = 2$).

Hence, from (3.12) it follows that this point is also in the set bounded by hyperplanes (3.16).

Therefore we have the inequalities

$$(3.17)\qquad \begin{cases} r_l^M \leqslant r_l^{A_l} \leqslant Y_{l2} < \dfrac{-\varepsilon(h)}{\eta+(p-1)\cdot\delta} < \dfrac{-\varepsilon(h)}{\eta}, \\ (l = 1, \ldots, p)(p \geqslant 2). \end{cases}$$

In the similar manner we prove that

$$(3.18)\qquad \begin{cases} r_l^M \geqslant r_l^{B_l} \geqslant Y_{l2} > \dfrac{\varepsilon(h)}{\eta+(p-1)\delta} > \dfrac{\varepsilon(h)}{\eta}, \\ (l = 1, \ldots, p)(p \geqslant 2). \end{cases}$$

From (3.17) and (3.18) we obtain the desired error estimate (3.1) and the inequalities (3.2).

This completes the proof of the Theorem 1.

References

[1] Z. Kowalski, *The Neumann problem for a system of non-linear elliptic equations*, Universitatis Iagellonicae Acta Mathematica, this issue, 95—108.

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